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Abstract

Full Text

Mathematics

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The Area Principle in the Problem of Non-Overlapping Domains

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Let $a_k,\ k = 0, 1, \dots, n;\ n = 0, 1, \dots,$ be given distinct points of the extended z-plane. Let D_k $(a_k \in D_k),\ k = 0, 1, \dots, n,$ be arbitrary simply connected domains of the extended z-plane having no pairwise common points. Denote by $w = f_k(z),\ f_k(0) = a_k,\ k = 0, 1, \dots, n,$ a function mapping the disk $|z| < 1$ conformally and univalently onto the domain D_k. In this way a system $\{f_k(z)\}_0^n$ of $n + 1$ functions is obtained. The set of all such systems of functions will be called the class $\mathfrak{M}(a_0, a_1, \dots, a_n)$.

Let: R be the class of functions $w = f(z) = \sum_{k=1}^{\infty} a_k z^k$, regular in the disk $|z| < 1$ and such that, for any points z_1 and z_2 in $|z| < 1$, the product $f(z_1)f(z_2) \neq 1$; Γ be the class of functions $w = f(z) = \sum_{k=1}^{\infty} a_k z^k$, regular in the disk $|z| < 1$ and such that, for any points z_1 and z_2 in $|z| < 1$, the product $f(z_1)\overline{f(z_2)} \neq -1$; R^* and Γ^* are the subclasses of univalent functions respectively from the classes R and Γ.

Let $\{f_k(z)\}_0^n \in \mathfrak{M}(\infty, a_1, \dots, a_n)$. Denote by $D_k(r)$, $k = 0, 1, \dots, n$, the image of the disk $|z| < r,\ 0 < r < 1$, under the mapping by the function $w = f_k(z)$, and by $D(r)$ the complement in the extended w-plane of $\bigcup_{k=0}^{n} \overline{D_k(r)}$.

1°. Let $\{f_k(z)\}_0^n \in \mathfrak{M}(\infty, a_1, \dots, a_n)$. Let the function $\xi = Q(w)$ be regular (and single-valued) in the domain $D(r_0)$ for some $r_0,\ 0 < r_0 < 1$, and consequently the functions $Q(f_l(z))$ are regular in the annulus $r_0 < |z| < 1$ and are representable there in the form

$$Q(f_l(z)) = \sum_{q=1}^{\infty} \frac{\beta_q^{(l)}}{z^q} + \sum_{q=0}^{\infty} b_q^{(l)} z^q.$$

Lemma. The inequality holds

$$\sum_{l=0}^{n} \sum_{q=1}^{\infty} q \left|b_q^{(l)}\right|^2 \le \sum_{l=0}^{n} \sum_{q=1}^{\infty} q \left|\beta_q^{(l)}\right|^2 = A. \tag{1}$$

To prove this inequality it is necessary to compute the area $S(r)$ of the image of the domain $D(r)$, $r_0 < r < 1$, under the mapping by the function $\xi = Q(w)$ (see [1,2]) and let r tend to one. Equality in (1) holds if and only if $\lim_{r\to 1} S(r) = 0$.

If, in the statement of the lemma, one requires only that the function $Q(w)$ have a regular and (single-valued) derivative in $D(r_0)$, then in the right-hand side of inequality (1) there will appear some additional term, owing to the fact that in the expansion of the function $Q(f_l(z))$ in the annulus $r_0 < |z| < 1$ there will appear a term $\beta^{(l)} \ln z$.

Corollary. Let $C_q^{(l)}$ be arbitrary numbers such that the series $\sum_{q=1}^{\infty} q|C_q^{(l)}|$, $l = 0, 1, \dots, n$, converge. If the conditions of the lemma are satisfied, the following inequalities hold:

$$\sum_{l=0}^{n} \lambda_l \left|\sum_{q=1}^{\infty} q b_q^{(l)} C_q^{(l)}\right| \le A - \sum_{l=0}^{n}\sum_{q=1}^{\infty} q \left|b_q^{(l)} - \lambda_l e^{i\theta_l} C_q^{(l)}\right|^2, \tag{2}$$

where

$$\lambda_l = \left|\sum_{q=1}^{\infty} q b_q^{(l)} C_q^{(l)}\right| \cdot \left(\sum_{q=1}^{\infty} q|C_q^{(l)}|^2\right)^{-1}, \qquad \theta_l = \arg \sum_{q=1}^{\infty} q b_q^{(l)} C_q^{(l)};$$

$$\left(\sum_{l=0}^{n} \left|\sum_{q=1}^{\infty} q b_q^{(l)} C_q^{(l)}\right|\right)^2 \le \left(A - \sum_{l=0}^{n}\sum_{q=1}^{\infty} q \left|b_q^{(l)} - \lambda e^{i\theta_l} C_q^{(l)}\right|^2\right) \sum_{l=0}^{n}\sum_{q=1}^{\infty} q|C_q^{(l)}|^2, \tag{3}$$

where

$$\lambda = \left(\sum_{l=0}^{n} \left|\sum_{q=1}^{\infty} q b_q^{(l)} C_q^{(l)}\right|\right) \cdot \left(\sum_{l=0}^{n}\sum_{q=1}^{\infty} q|C_q^{(l)}|^2\right)^{-1}, \qquad \theta_l = \arg \sum_{q=1}^{\infty} q b_q^{(l)} C_q^{(l)}.$$

Equality in inequalities (2) and (3) holds if and only if $\lim_{r\to 1} S(r) = 0$.

2°. Let $z_{\nu,k}, z'_{\nu,k}$ ($\nu = 0, 1, \dots, m;\ k = 0, 1, \dots, n$) be arbitrary points from the disk $|z| < 1$; let $\gamma_{\nu,k}, \gamma'_{\nu,k}$ ($\nu = 0, 1, \dots, m;\ k = 0, 1, \dots, n$), $x_{\nu,k}, x'_{\nu,k}$ ($\nu = 0, 1, \dots, m,\ k = 0, 1, \dots, n$) be arbitrary numbers.

Let $\{f_k(z)\}_0^n \in \mathfrak{M}(\infty, a_1, \dots, a_n)$. Consider functions $\psi_k(w, \xi)$, $k = 0, 1, \dots, n$, such that the function $\psi_k(w, \xi)$, for every fixed $\xi \in D_k(r)$, $0 < r < 1$, is regular in w outside $\overline{D}_k(r)$ and, for every fixed $w \in \overline{D}_k(r)$, is regular in $D_k(r)$ with respect to ξ. It is clear that for $k \neq l$

$$\psi_k(f_l(z), f_k(\xi)) = \sum_{q=0}^{\infty}\sum_{p=0}^{\infty} b_{p,q}^{k,l}\xi^p z^q = \sum_{q=0}^{\infty} b_q^{k,l}(\xi) z^q = b_0^{k,l}(\xi) + \varphi_{k,l}(\xi, z)$$

for $|z| < 1,\ |\xi| < 1$;

$$\psi_l(f_l(z), f_l(\xi)) = \sum_{q=0}^{\infty}\sum_{p=0}^{\infty} b_{p,q}^{l,l}\xi^p z^q + \sum_{q=1}^{\infty}\sum_{p=0}^{\infty} \beta_{p,q}^{l,l}\xi^p z^{-q} =$$

$$= \sum_{q=0}^{\infty} b_q^{l,l}(\xi) z^q + \sum_{q=1}^{\infty} \beta_q^{l,l}(\xi) z^{-q} = b_0^{l,l}(\xi) + \varphi_{l,l}(\xi, z) + \sum_{q=1}^{\infty} \beta_q^{l,l}(\xi) z^{-q}$$

in the domain $|\xi| < |z| < 1$.

Introduce the function

$$Q(w) = \sum_{k=0}^{n}\sum_{\nu=0}^{m} \gamma_{\nu,k}\psi_k(w, f_k(z_{\nu,k})).$$

This function satisfies the conditions of the lemma. Consequently, the following inequalities hold:

$$\sum_{l=0}^{n}\sum_{q=1}^{\infty} q\left|\sum_{k=0}^{n}\sum_{\nu=0}^{m}\gamma_{\nu,k} b_q^{k,l}(z_{\nu,k})\right|^2 \le \sum_{l=0}^{n}\sum_{q=1}^{\infty} q\left|\sum_{\nu=0}^{m}\gamma_{\nu,l}\beta_q^{l,l}(z_{\nu,l})\right|^2 = A'; \qquad (1')$$

$$\sum_{l=0}^{n}\frac{1}{A'_l}\left|\sum_{k=0}^{n}\sum_{\nu,\nu'=0}^{m}\gamma_{\nu,k}\gamma'_{\nu',l}\frac{\partial^s}{\partial z_{\nu,k}^s}\varphi_{k,l}(z_{\nu,k}, z'_{\nu',l})\right| \le A', \qquad s = 0, 1, \dots; \qquad (2')$$

$$\sum_{l=0}^{n}\left|\sum_{k=0}^{n}\sum_{\nu,\nu'=0}^{m}\gamma_{\nu,k}\gamma'_{\nu',l}\frac{\partial^s}{\partial z_{\nu,k}^s}\varphi_{k,l}\left(z_{\nu,k}, z'_{\nu',l}\right)\right|^2 \le \left[A'\sum_{l=0}^{n}A'_l\right]^{1/2}, \qquad s = 0, 1, \dots, \qquad (3')$$

where

$$A'_l = \sum_{\nu,\nu'=0}^{m}\gamma'_{\nu,l}\overline{\gamma'_{\nu',l}}\left[\frac{\partial^{2s}}{\partial z^s\partial\zeta^s}\ln\frac{1}{1-z\zeta}\right]_{\zeta=z'_{\nu,l};\, z=\overline{z'_{\nu',l}}}.$$

To obtain inequalities $(2')$ and $(3')$, one must apply the corollary of the lemma to inequality $(1')$, putting

$$C_q^{(l)} = \frac{1}{q}\sum_{\nu=0}^{m}\gamma'_{\nu,l}\,\frac{q!}{(q-s)!}\,(z'_{\nu,l})^{q-s}$$

for $q \geq s$, and $C_q^{(l)} = 0$ for $q < s$.

Putting in inequalities (1′), (2′), and (3′)

$$\gamma_{\nu,k} = \frac{1}{\overline{m}+1}\sum_{p=0}^{\overline{m}} x_{p,k}\overline{z}_{\nu,k}^{p},$$

$$\gamma'_{\nu,l} = \frac{1}{\overline{m}+1}\sum_{q=0}^{\overline{m}} x'_{q,l}(\overline{z}_{\nu,l})^{q}, \qquad z_{\nu,l} = z'_{\nu,l} = re^{i\theta_\nu}, \qquad \theta_\nu = \frac{2\pi}{m+1}\nu, \quad 0 < r < 1,$$

and letting m tend to ∞, and then r to unity, in the limit we obtain the inequalities

$$\sum_{l=0}^{n}\sum_{q=1}^{\infty} q\left|\sum_{k=0}^{n}\sum_{p=0}^{\overline{m}} b_{p,q}^{k,l}x_{p,k}\right|^2 \leq \sum_{l=0}^{p}\sum_{q=1}^{\infty} q\left|\sum_{p=0}^{\overline{m}} \beta_{p,q}^{l,l}x_{p,l}\right|^2 = A'', \qquad (1")$$

$$\sum_{l=0}^{n}\frac{1}{A''_l}\left|\sum_{k=0}^{n}\sum_{p,q=0}^{\overline{m}} b_{p,q}^{k,l}\frac{(q+s)!}{q!}\,x_{p,k}x'_{q,l}\right|^2 \leq A'', \qquad s = 0, 1, \dots; \qquad (2")$$

$$\sum_{l=0}^{n}\left|\sum_{k=0}^{n}\sum_{p,q=0}^{\overline{m}} b_{p,q+s}^{k,l}\frac{(q+s)!}{q!}\,x_{q,k}x'_{q,l}\right| \leq \left[A''\sum_{l=0}^{n}A''_0\right]^{1/2}, \qquad s = 0, 1, \dots, \quad (3")$$

where

$$A''_l = \sum_{q=0}^{\overline{m}}\frac{(q+s)!}{q!}\,\frac{1}{q}\,|x'_{q,l}|^2.$$

In inequalities (2″) and (3″), for $s = 0$ the summation over q must run from 1 to $\overline{m}$ (and not from zero to $\overline{m}$).

Putting in inequalities (1″), (2″), and (3″)

$$x_{p,k} = \sum_{\nu=0}^{m}\gamma_{\nu,k}z_{\nu,k}^{p}, \qquad x'_{q,l} = \sum_{\nu=0}^{m}\gamma'_{\nu,l}(z'_{\nu,l})^{q}$$

and letting $\overline{m}$ tend to ∞, in the limit we obtain inequalities (1′), (2′), and (3′).

3°. As the functions $\psi_k(w,\xi)$ occurring in item 2°, one may take, for example,

$$\psi_k(w,\xi) = \begin{cases} \ln\left(1 - \dfrac{\xi - a_k}{w - a_k}\right), & k = 1, 2, \dots, n, \\ \ln\left(1 - \dfrac{w}{\xi}\right), & k = 0. \end{cases}$$

In this case

$$A' = \sum_{l=0}^{n} \sum_{\nu,\nu'=0}^{m} \gamma_{\nu,l}\overline{\gamma_{\nu',l}} \ln \frac{1}{1 - z_{\nu,l}\overline{z}_{\nu',l}} = \sum_{l=0}^{n} \sum_{\nu,\nu'=0}^{m} \gamma_{\nu,l}\overline{\gamma_{\nu',l}} \sum_{q=1}^{\infty} \frac{1}{q}(z_{\nu,l}\overline{z}_{\nu',l})^q,$$

$$A'' = \sum_{l=0}^{n} \sum_{q=1}^{m} \frac{1}{q}|x_{q,l}|^2.$$

4°. In the case considered in item 3°, the inequalities (1′), (2′), (3′), (1″), (2″), (3″) can be generalized. In this case, for example, inequality (2) (for $s = 0$) will correspond to the inequality

$$\sum_{l=0}^{n} \frac{1}{A_l^{(s)}} \left| \sum_{k=0}^{n} \sum_{\nu,\nu'=0}^{m} \gamma'_{\nu,k}\bar{\gamma}_{\nu',l}\, \varphi_{k,l}^{s}(z'_{\nu,k}, z_{\nu',l}) \right|^2 \le \sum_{l=0}^{n} \sum_{\nu,\nu'=0}^{m} \gamma'_{\nu,l}\bar{\gamma}_{\nu',l} \ln^s \frac{1}{1 - z'_{\nu,l}\bar{z}_{\nu',l}}, \qquad s = 1, 2, \dots,$$

where

$$A_l^{(s)} = \sum_{\nu,\nu'=0}^{m} \gamma'_{\nu,l}\bar{\gamma}_{\nu',l} \ln^s \frac{1}{1 - z'_{\nu,l}\bar{z}_{\nu',l}}.$$

Further, it is easy to obtain six more inequalities. In this case, to the preceding inequality there will correspond the inequality

$$\sum_{l=0}^{n} \frac{1}{A_l^*} \left| \sum_{k=0}^{m} \sum_{\nu,\nu'=0}^{m} \gamma'_{\nu,k}\bar{\gamma}_{\nu',l}\, e^{\varphi_{k,l}(z_{\nu,k}, z'_{\nu',l})} \right|^2 \le \sum_{l=0}^{n} \sum_{\nu,\nu'=0}^{m} \frac{\gamma'_{\nu,l}\bar{\gamma}_{\nu',l}}{1 - z'_{\nu,l}\bar{z}'_{\nu',l}},$$

$$A_l^* = \sum_{\nu,\nu'=0}^{m} \frac{\gamma'_{\nu,l}\bar{\gamma}_{\nu',l}}{1 - z'_{\nu,l}\bar{z}'_{\nu',l}}.$$

The latter inequalities make it possible to obtain a number of interesting integral inequalities. In particular, the following holds.

Theorem. If $\{f_0(z), f_1(z)\} \in \mathfrak{M}(\infty, 0)$, then the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} |f_1(e^{i\theta})|^2\, d\theta \cdot \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{|f_0(e^{i\theta})|^2}\, d\theta \le 1.$$

holds.

The equality sign holds if and only if

$$f_0(z) = \frac{a}{z} + b, \qquad |a| > |b|, \qquad f_1(z) = \frac{(|a|^2 - |b|^2)\eta z}{\bar{a} - \bar{b}\eta z}, \qquad |\eta| = 1.$$

Corollary. If $f(z) = \sum_{k=1}^{\infty} a_k z^k \in R^*$ (or Γ^*), then the inequality

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta = \sum_{k=1}^{\infty} |\alpha_k|^2 \le 1. \tag{4}$$

holds.

The equality sign holds if and only if

$$f(z) = \frac{\eta z}{R \pm \sqrt{R^2 - 1}\,\eta z}, \qquad R \ge 1, \qquad |\eta| = 1.$$

Inequality (4) also holds in the case when $f(z) \in R$ (or Γ). It strengthens the result obtained in paper (1):

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|\, d\theta \le 1, \qquad f(z) \in R \text{ (or } \Gamma).$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.