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Abstract

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ON ONE SUFFICIENT CRITERION FOR HÖLDER CONTINUITY OF A MAPPING

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Let $\mathbf{y}(\mathbf{x}) = (y_1(\mathbf{x}), y_2(\mathbf{x}), \dots, y_m(\mathbf{x}))$ be a continuous mapping of an open domain M in n-dimensional Euclidean space E^n into m-dimensional space E^m, $m \geq n$. We say that $\mathbf{y}(\mathbf{x}) \in W_p^l(M)$ if each of the functions $y_k(\mathbf{x})$, $k = 1, 2, \dots, m$, has in M generalized partial derivatives of order l, in the sense of S. L. Sobolev (^{1}), summable on every compact set $F \subset M$. We put

$$\lambda[\mathbf{x}; \mathbf{y}] = \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\frac{\partial y_i(\mathbf{x})}{\partial x_j} \right]^2 .$$

Let $Q(\mathbf{x}_0, r)$ be the ball $\|\mathbf{x} - \mathbf{x}_0\| < r$, and $S(\mathbf{x}_0, r)$ the sphere $\|\mathbf{x} - \mathbf{x}_0\| = r$ in E^n. For an arbitrary measurable function $\lambda(\mathbf{x}) \geq 0$ we put

$$V_\lambda(\mathbf{x}, r) = \iint_{Q(\mathbf{x}, r)} [\lambda(\mathbf{x})]^{n/2} \, dx_1 \dots dx_n,$$

$$F_\lambda(\mathbf{x}, r) = \int_{S(\mathbf{x}, r)} [\lambda(\mathbf{x})]^{(n-1)/2} \, d\sigma,$$

where $d\sigma$ is the element of area of the sphere $S(\mathbf{x}, r)$.

A mapping $\mathbf{y}(\mathbf{x})$ of the domain $M \subset E^n$ is called a mapping of class $I(k)$, $0 < k < 1$, if for every point $\mathbf{x} \in M$, for $0 < r \leq r_0(\mathbf{x})$, where $r_0(\mathbf{x})$ is the distance from $\mathbf{x}$ to the boundary of M, the inequality

$$[F_\lambda(\mathbf{x}, r)]^n \geq (nk)^{n-1} \omega_{n-1} [V_\lambda(\mathbf{x}, r)]^{n-1}, \tag{1}$$

holds, where $\lambda(\mathbf{x}) = \lambda(\mathbf{x}; \mathbf{y})$, and ω_{n-1} is the area of the sphere $S(0, 1)$.

Theorem 1. If the mapping $\mathbf{y}(\mathbf{x}) \in I(k)$, then for every compact set $F \subset M$, when $\mathbf{x}_1 \in F$, $\mathbf{x}_2 \in F$,

$$\|\mathbf{y}(\mathbf{x}_1) - \mathbf{y}(\mathbf{x}_2)\| \leq A \|\mathbf{x}_1 - \mathbf{x}_2\|^k,$$

where the quantity A depends only on $d = \inf_{x \in F} r_0(\mathbf{x})$ and on

$$\|\lambda\|_{L_{n/2}(F)} = \left(\iint_F [\lambda(\mathbf{x})]^{n/2} \, dx_1 \ldots dx_n \right)^{2/n} .$$

Proof. Applying Hölder' s inequality, we obtain

$$F_\lambda(\mathbf{x}, r) \leq \left(\int_{S(\mathbf{x},r)} [\lambda(\mathbf{x})]^{n/2} \, d\sigma \right)^{(n-1)/n} \left(\int_{S(\mathbf{x},r)} d\sigma \right)^{1/n} = \left(\frac{dV_\lambda(\mathbf{x}, r)}{dr} \right)^{(n-1)/n} (\omega_{n-1} r^{n-1})^{1/n}. \tag{2}$$

Hence, substituting (2) into (1), we obtain:

$$r \frac{dV_\lambda(x, r)}{dr} - nkV_\lambda(x, r) \geqslant 0. \tag{3}$$

Multiplying both sides of (3) by r^{-1-nk} and integrating with respect to r from r_1 to r_2, $0 < r_1 \leqslant r_2 < r_0(x)$, we obtain

$$\frac{V_\lambda(x, r_2)}{r_2^{nk}} - \frac{V_\lambda(x, r_1)}{r_1^{nk}} \geqslant 0.$$

It follows that for $0 < r < d$, where $d = \inf_{x \in F} r_0(x)$,

$$V_\lambda(x, r) \leqslant \frac{V_\lambda(x, d)}{d^{nk}} r^{nk} \leqslant A r^{nk}, \tag{4}$$

where A depends only on d and on $\|\lambda\|_{L^{n/2}(F)}$. On the basis of Morrey' s theorem (^{2}), the assertion of the theorem follows from this.

Let us note some applications of Theorem 1. A. G. Sigalov (^{3}) proved a theorem on the existence of a minimum in the class of continuous surfaces $x(u, v)$ belonging to $W_2^1(Q)$, for two-dimensional functionals of the calculus of variations of the form

$$J(x, Q) = \iint_Q F(x, x_u \times x_v) \, du \, dv, \tag{5}$$

where Q is the square $(0 \leqslant u \leqslant 1,\ 0 \leqslant v \leqslant 1)$; the function F satisfies certain conditions, in particular, for all (x, a)

$$0 < m\|a\| \leqslant F(x, a) \leqslant M\|a\|$$

(m and M are constants). In the course of the proof of this theorem, A. G. Sigalov established that the surface realizing the minimum, in our terminology, belongs to the class $I\left(\frac{m}{M}\right)$.

Hence, on the basis of Theorem 1, we obtain the following theorem.

Theorem 2. If a function $x(u,v) \in W_2^1(Q)$ realizes the minimum of the functional (5) under the conditions of A. G. Sigalov' s theorem, then it satisfies the Hölder condition inside Q with exponent $k = m/M$.

Let $m = n$. A mapping $y(x)$ is called **quasiconformal** if:

a) $y(x)$ is a topological mapping;

b) the vector function $y(x)$ has continuous first derivatives $\partial y(x)/\partial x^k$, $k = 1, 2, \dots, n$, and the Jacobian of the mapping

$$J(x) = \frac{\partial(y_1, y_2, \dots, y_n)}{\partial(x_1, x_2, \dots, x_n)} > 0 \tag{6}$$

for all $x \in M$;

c) there exists a constant $q \geqslant 1$ such that, for each point $x \in M$, the mapping $y(x)$ transforms an infinitesimal sphere with center x into an infinitesimal ellipsoid whose ratio of the greatest and least semiaxes does not exceed q.

The number q is called the **coefficient of the quasiconformal mapping**. For quasiconformal mappings, as is known, the inequality

$$[\lambda(x;y)]^{n/2} \leqslant [k(q)]^{n-1} J(x), \tag{7}$$

holds, where $k(q) < \infty$ depends only on q.

Lemma. Every quasiconformal mapping belongs to the class $I[k(q)]$.

Proof. Introduce in M a metric with line element

$$dS^2 = dy^2 = \sum_{i,j} g_{ij}\, dx_i\, dy_j, \qquad g_{ij} = \frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j}; \tag{8}$$

this metric is Euclidean. The area $F_y(S)$ and the volume $V_y(Q)$ of the interior domain Q, measured in the metric (8) for a smooth closed surface $S \subset M$, satisfy the isoperimetric inequality

$$[F_y(S)]^n \geq n^{n-1} \omega_{n-1} [V_y(Q)]^{n-1}. \tag{9}$$

Let us note that the volume $V_y(Q)$ of the domain Q is equal to the integral of the Jacobian $J(x)$ over the domain Q.

Consider in M the metric

$$dS_1^2 = \lambda(x;y)(dx_1^2 + \dots + dx_n^2). \tag{10}$$

For all x and arbitrary $dx_1, \dots, dx_n$,

$$dS_1^2 \geq dS^2.$$

Consequently, the area of the surface S in the metric (10) is not less than its area in the metric (8), i.e.

$$\int_S [\lambda(x)]^{(n-1)/2}\, d\sigma \geq F_y(S),$$

where $d\sigma$ is the element of area in the ordinary Euclidean metric. On the basis of (9), it follows from this that

$$\left\{\int_S [\lambda(x)]^{(n-1)/2}\, d\sigma\right\}^n \geq n^{n-1}\omega_{n-1}\left\{\iint_Q J(x,y)\, dx\right\}^{n-1} \geq$$

$$\geq n^{n-1}\omega_{n-1}[k(q)]^{n-1}\left(\iint_Q [\lambda(x)]^{n/2}\, dx\right)^{n-1}.$$

The last inequality is obvious, and means that $y(x) \in I(k)$.

From the lemma, on the basis of Theorem 1, the following theorem follows.

Theorem 3. Quasiconformal mappings with quasiconformality coefficient q of the ball $Q\{|x| < 1\}$ onto itself, on every compact set $F \subset Q(0,1)$, satisfy the Hölder condition with an exponent depending only on q, and with a coefficient depending only on q, F.

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