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Abstract

Full Text

GEOPHYSICS

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ON A NONLINEAR THEORY OF UNSTEADY SLOPE WIND

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In the present work we consider the plane unsteady problem of a wind arising over an inclined thermally homogeneous surface. Introduce a curvilinear coordinate system (x, z) associated with the relief (the x -axis is directed along the line of relief, z upward). For our problem a typical system of equations will be (1)

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} &= \lambda \sin \alpha \vartheta + \frac{\partial}{\partial z} \nu \frac{\partial u}{\partial z} \quad \left(\lambda = \frac{g}{T} \right); \\ \frac{\partial \vartheta}{\partial t} + u \frac{\partial \vartheta}{\partial x} + w \frac{\partial \vartheta}{\partial z} &= -\mu \sin \alpha u + \frac{\partial}{\partial z} \chi \frac{\partial \vartheta}{\partial z} \quad \left(\mu = \gamma_a + \frac{dT}{dz} \right); \\ \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} &= 0. \end{aligned} \quad (1)$$

Here t is time; u, w are the components of the wind velocity, respectively, along the axes x, z ; ϑ is the deviation of the temperature from T , its value in the resting atmosphere; g is the acceleration due to gravity; γ_a is the dry-adiabatic gradient; $\alpha = \alpha(x)$ is the angle of inclination of the lines of relief (the slope) to the horizontal surface; ν, χ are the coefficients of vertical turbulent viscosity and thermal conductivity, respectively.

Put $\chi = \nu = \text{const}$. The conditions of the problem will be

$$\begin{aligned} u = \vartheta = 0 \quad \text{for } t = 0 \text{ and for } z = \infty; \\ u = w = 0, \quad \vartheta = f(t) \quad \text{for } z = 0, \end{aligned} \quad (2)$$

where $f(t)$ is the prescribed temperature of the slope surface.*

Let us now proceed to the solution of the problem, first putting $\lambda = \text{const}$, $\mu = \text{const}$. In order to reduce the problem to ordinary differential equations, we shall use a device analogous to that employed by A. A. Dorodnitsyn.

Denote

$$\xi = \frac{x}{L}, \quad \zeta = \frac{1}{2} \sqrt[4]{\frac{\lambda \sin \alpha_0 \theta_{00}}{\nu^2 L}} \frac{z}{\tau}, \quad \tau = \sqrt[4]{\frac{\lambda \sin \alpha_0 \theta_{00}}{L}} \sqrt{t},$$

where L, α_0 , and θ_{00} are, respectively, a characteristic length, characteristic slope, and characteristic value of the slope temperature.

If now $f(t)$ is represented in the form

$$f(t) = \theta_{00} F(t) = \theta_{00} \sum_{n=0}^{\infty} F_n \tau^{n+2}, \quad (3)$$

* The solution obtained below can easily be generalized to the case when $\vartheta_{z=0}$, being a function of the coordinates and time, is determined from the condition of heat balance at the slope surface.

where F_n are numerical coefficients, one may seek the solution of system (1) in the form*

$$\vartheta = \theta_{00} \sum_{n=0}^{\infty} \theta_n \tau^{n+2}, \quad u = \sqrt{\lambda \sin \alpha_0 L \theta_{00}} \sum_{n=0}^{\infty} U_n \tau^{n+4}, \quad (4)$$

$$w = 2 \sqrt[4]{\frac{\nu^2}{L} \lambda \sin \alpha_0 \theta_{00}} \sum_{n=0}^{\infty} W_n \tau^{n+5},$$

where θ_n, U_n , and W_n are functions depending on ξ and ζ and satisfying the infinite system

$$\Lambda_{n+4}(U_n) = -4\chi\theta_n + 4 \sum_{k=0}^{n-6} \left(U_k \frac{\partial U_{n-6-k}}{\partial \xi} + W_k \frac{\partial U_{n-6-k}}{\partial \zeta} \right); \quad (5)$$

$$\Lambda_{n+2}(\theta_n) = 4\Gamma\chi U_{n-4} + 4 \sum_{k=0}^{n-6} \left(U_k \frac{\partial \theta_{n-6-k}}{\partial \xi} + W_k \frac{\partial \theta_{n-6-k}}{\partial \zeta} \right); \quad (6)$$

$$\frac{\partial U_n}{\partial \xi} + \frac{\partial W_n}{\partial \zeta} = 0, \quad (7)$$

where the notation** is

$$\Lambda_m(y_n) = y_n'' + 2\zeta y_n' - 2m y_n, \quad \chi = \frac{\sin \alpha}{\sin \alpha_0}, \quad \Gamma = \frac{\mu}{\theta_{00}} \sin \alpha_0 L. \quad (8)$$

The system (5), (6), (7) must be solved under the boundary conditions

$$U_n = \theta_n = 0 \quad \text{for } \xi = \infty; \quad U_n = W_n = 0, \quad \theta_n = F_n \quad \text{for } \xi = 0, \quad (9)$$

which are a consequence of (2).

From (6), for $n < 4$, we obtain for θ_n the homogeneous equation

$$\Lambda_{n+2}(\theta_n) = \theta_n'' + 2\zeta\theta_n' - 2(n+2)\theta_n = 0. \quad (10)$$

Under the boundary conditions (9), equation (10), as is known (2), has the solution

$$\theta_n = F_n L_{n+2}(\zeta). \quad (11)$$

The functions L_m were introduced in (3) and have the form

$$L_m(\zeta) = \frac{A_m}{m!} \int_{\infty}^{\zeta} (\zeta - \zeta')^m \exp(-\zeta'^2) d\zeta', \quad (12)$$

$$(A_m = 2mA_{m-2}, \quad A_0 = -2/\sqrt{\pi}, \quad A_1 = 2, \quad L_m(0) = 1, \quad L_m(\infty) = 0).$$

Next, up to $n < 6$, for θ_n and U_n one must solve an inhomogeneous equation of the form

$$\Lambda_m(y_n) = \sum_{k=0}^r a_k L_k, \quad (13)$$

* The convergence of the series depends on the decrease of the coefficients θ_n , U_n , and W_n and is not rigorously justified by us. However, the series for small moments of time (at least up to $\tau \approx 1$) apparently converge asymptotically.

** A prime denotes derivatives with respect to ζ . We shall see that the system (5), (6), (7) depends on ξ only parametrically; therefore in the left-hand sides of equations (5) and (6) we have an ordinary linear operator.

where the coefficients a_k depend on ξ . Taking into account that

$$\Lambda_m(a_k L_k) = 2a_k(k-m)L_k,$$

we obtain the solution of (13), under the boundary conditions (9), in the form

$$y_n = F_n L_m + \sum_{k=0}^r \frac{a_k}{2(k-m)} (L_k - L_m). \quad (14)$$

Beginning with $n \geq 6$, nonlinear terms appear in (5) and (6). For the problem under consideration, the right-hand side of the nonhomogeneous equation (for $n \geq 6$) will be combinations of functions L_m with lower indices. For example, beginning with $n = 6$ up to $n = 11$, it is necessary to solve a nonhomogeneous equation of the form

$$\Lambda_m(y_n) = \sum_{i,j} b_{i,j} L_i L_j, \quad (15)$$

where $b_{i,j}$ depend on ξ .

Using the properties of the functions L_m , one can establish that

$$\Lambda_m \left(\frac{b_{i,j}}{2} \left[\frac{A_i A_j}{A_{i+1} A_{j+1}} L_{i+1} L_{j+1} + \sum_{k=2}^r 2^{k-1} (r-1)(r-2) \dots (r-k+1) \frac{A_i A_j}{A_{i+k} A_{j+k}} L_{i+k} L_{j+k} \right] \right) = b_{i,j} L_i L_j,$$

where $* 2r = m - (i + j)$.

Then the solution of (15), under the boundary conditions (9), will be

$$y_n = F_n L_m + \sum_{i,j} \frac{b_{i,j}}{2} \left\{ \frac{A_i A_j}{A_{i+1} A_{j+1}} (L_{i+1} L_{j+1} - L_m) + \sum_{k=2}^r 2^{k-1} (r-1)(r-1) \dots (r-k+1) \frac{A_i A_j}{A_{i+k} A_{j+k}} (L_{i+k} L_{j+k} - L_m) \right\}. \quad (16)$$

Thus, using the solutions (11), (14), and (16), one can determine θ_n , U_n , and W_n up to and including $n = 11^{**}$. In order not to encumber the calculations, we shall consider the solution of the system (5), (6), (7), under the boundary conditions (9), for the first 10 indices ($n \leq 9$). Then we finally obtain

$$\theta_n = F_n \theta_{n,0} - \Gamma \chi^2 F_{n-4} \theta_{n,1} + \Gamma^2 \chi^4 F_{n-8} \theta_{n,2} - \sum_{k=0}^{n-6} F_{n-6-k} F_k \frac{\partial \chi}{\partial \xi} \theta_{n,k+3}; \quad (17)$$

$$U_n = \chi \left\{ F_n U_{n,0} - \Gamma \chi^2 F_{n-4} U_{n,1} + \Gamma^2 \chi^4 F_{n-8} U_{n,2} - \sum_{k=0}^{n-6} F_{n-6-k} F_k \frac{\partial \chi}{\partial \xi} U_{n,k+3} \right\}; \quad (18)$$

$$W_n = -\frac{\partial \chi}{\partial \xi} \{ F_n W_{n,0} - 3\Gamma \chi^2 F_{n-4} W_{n,1} + 4\Gamma^2 \chi^3 F_{n-8} W_{n,2} \} +$$

Fig. 1 and Fig. 2: curves of the functions described below.

Figure 1: Fig. 1 and Fig. 2: curves of the functions described below.

$$+ \sum_{k=0}^{n-6} F_{n-6-k} F_k \frac{\partial}{\partial \xi} \left(\chi \frac{\partial \chi}{\partial \xi} \right) W_{n,k+3} \quad (F_{-n} \equiv 0, n \leq 9), \quad (19)$$

* We note that for the problem under consideration the condition $m-(i+j) = 2r$ is always satisfied.

** The determination of further terms of the expansions presents no fundamental or technical difficulties if th

where $\theta_{n,j}$, $U_{n,j}$, and $W_{n,j}$ ($j = 0, \dots, 6$) are functions of only one argument ζ , which are combinations of the functions L_m ; they are universal in character and can be calculated in advance. The expressions for these functions as functions of ζ have been obtained by us explicitly; we do not present them here for lack of space. All these functions have been tabulated by us. Note that all $\theta_{n,j}$ and $U_{n,j}$ tend to zero both for $\zeta = 0$

Fig. 1. Functions $\theta_{n,j}$ as functions of ζ

Fig. 2. Functions $U_{n,j}$ and $W_{n,j}$ as functions of ζ

(except for $\theta_{n,0}$, which at $\zeta = 0$ is equal to unity), and for $\zeta \rightarrow \infty$. $W_{n,j} = 0$ for $\zeta = 0$, while for $\zeta \rightarrow \infty$ all $W_{n,j}$ assume various constant values, as follows from the conditions of our problem. Graphs of some of the functions $\theta_{n,j}$, $U_{n,j}$, and $W_{n,j}$ are given in Figs. 1 and 2.

Using the formulas given above, we have calculated examples describing the development of a slope wind over time and with height at various points of the slope.

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