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Abstract

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INTEGRAL EQUATIONS WITH ALMOST DIFFERENCE KERNELS

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1. By integral equations with almost difference kernels we shall mean equations of the form

$$a(t)\varphi(t) + \int_{-\infty}^{\infty} k(t, t-\tau)\varphi(\tau)\, d\tau = f(t). \tag{1}$$

If, for any c, the convolution theorem is applicable to the pair $k(c,t)$ and $\varphi(t)$, where $\varphi(t)$ is a finite infinitely differentiable function, then the operator (1) can be represented in the form

$$k\varphi = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(t,\lambda)\Phi(\lambda)\exp(-i\lambda t)\, d\lambda, \tag{2}$$

where $\Phi(\lambda)$ is the Fourier transform of the function $\varphi(t)$, and

$$K(t,\lambda) = a(t) + \int_{-\infty}^{\infty} k(t,u)\exp(i\lambda u)\, du.$$

If, moreover, the operator (2) is bounded in the norm of the space $L_p(-\infty,\infty)$ on the set D, dense in it, of finite infinitely differentiable functions, then equation (1), considered in L_p, can be written in the form

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} K(t,\lambda)\Phi(\lambda)\exp(-i\lambda t)\, d\lambda = f(t). \tag{3}$$

The operator on the left-hand side of (3) is understood here as the extension of the operator (2) to all of $L_p(-\infty,\infty)$.

A special case of equation (3) was considered by I. M. Rapoport (^{1,2}) in the study of the Wiener-Hopf equation and paired equations. Multidimensional equations similar to equation (3) have been studied in a number of works by S. G. Mikhlin (for the bibliography, see (^{3})). Between the multidimensional equations considered in the works of S. G. Mikhlin and the one-dimensional equations of the form (3), despite a number of common features, there are essential differences, which will be apparent from what follows.

2. **Theorem 1.** *If for almost all t, $-\infty < t < \infty$: 1) the function $K(t, \lambda)$ is absolutely continuous in λ on any finite interval, there exists $\lim_{\lambda\to+\infty} K(t, \lambda) = K(t, +\infty)$, and $\sup_t |K(t, +\infty)| \le C$; 2) for some one-to-one and continuously differentiable mapping $\mu = \mu(\lambda)$ of the whole axis onto a finite segment the inequality*

$$\sup_t \int_{-\infty}^{\infty} \left|\frac{\partial K(t,\lambda)}{\partial\lambda}\right|^q |\mu'(\lambda)|^{q-1}\,d\lambda \le C_1, \qquad \frac{1}{p}+\frac{1}{q}=1,$$

holds, then the operator (2) is bounded in the norm of L_p on the set D.

Theorem 2. The assertion of the preceding theorem remains valid if condition 2) is replaced by the condition: $2'$) there exists a function $\Omega(\lambda) \in L$ such that for almost all t

$$|\partial K(t,\lambda)/\partial\lambda| \le \Omega(\lambda).$$

Theorem 3. Suppose that, for fixed t, the function $K(t, \lambda)$ has discontinuities of the first kind at the points of a fixed sequence $\{c_k\}$. If

$$\sum_k \sup_t |K(t, c_k+0) - K(t, c_k-0)| < \infty$$

and the function

$$K(t,\lambda) - \sum_k \theta(\lambda - c_k)[K(t, c_k+0) - K(t, c_k-0)]$$

satisfies the conditions of one of the preceding theorems, then the operator (2) is bounded in L_p ($\theta(\lambda)$ is the Heaviside function).

3. Let T be a completely continuous operator in L_p. Following S. G. Mikhlin, we shall call the function $K(t, \lambda)$ the **symbol of the operator**

$$A\varphi = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(t,\lambda)\Phi(\lambda)\exp(-i\lambda t)\,dt + T\varphi. \tag{4}$$

The definition of the symbol of an operator is not unique, since there exist completely continuous operators representable in the form (2), for example the operator with symbol $(1+\lambda^2)^{-1}e^{-|t|}$. From a given symbol the operator is recovered up to a completely continuous summand.

We now consider the totality of all operators whose symbols satisfy the conditions of Theorem 1 and the following conditions: the functions $K[t, \lambda(\mu)]$ and $\partial K[t, \lambda(\mu)]/\partial\mu$, where $\lambda(\mu) = -\operatorname{ctg}\frac{1}{2}\mu$, are continuous on the interval $[0, 2\pi]$, and each of them assumes equal values at the endpoints of this interval; the modulus of continuity $\omega(t, \delta)$ of the function $\partial K[t, \lambda(\mu)]/\partial\mu$ (with respect to the variable μ) tends uniformly with respect to t to zero as $\delta \to 0$; if the difference $t-\tau$ is sufficiently small in absolute value, or if $|t|$ and $|\tau|$ are sufficiently large and $t\tau > 0$, then

$$|K(t,\lambda) - K(\tau,\lambda)|\, e^{\mp i(t-\tau)} |t-\tau|^k\, \theta[\pm(t-\tau)] \leq$$

$$\leq \sum_{j=0}^{n_k} c_k^{(j)} |t-\tau|^{-\alpha_j} (1+t^2)^{\alpha_j/2-1/p} (1+\tau^2)^{\alpha_j/2-1/q},$$

where $0 \leq \alpha_j < 1$; $c_k^{(j)}$ are constants depending on the function $K(t,\lambda)$, and $k = 0, 1, \ldots$.

Theorem 4. The product of two operators of the form (4), whose symbols satisfy the conditions listed above, is an operator of the same form; moreover, the symbol of the product is equal to the product of the symbols of the factors.

Remark 1. If the function $K(t,\lambda)$ satisfies the conditions of Theorem 4, then the functions $\overline{K(t,\lambda)}$ and, if $\inf_{t,\lambda} |K(t,\lambda)| > 0$, $[K(t,\lambda)]^{-1}$ also satisfy these same conditions.

Remark 2. If the symbol $K(t,\lambda)$ of an operator A satisfies the conditions of Theorem 4 for the exponents p and q simultaneously, then the adjoint operator is representable in the form (4), and its symbol is equal to $\overline{K(t,\lambda)}$.

We introduce the following definitions:

We shall call a ring σ_1 the ring of bounded operators of the form (4) if: a) the assertion of Theorem 4 holds for this ring; b) if the ring contains an operator with symbol $K(t,\lambda)$, then it also contains the operators with symbols $\overline{K(t,\lambda)}$ and, if $\inf_{t,\lambda} |K(t,\lambda)| > 0$, $[K(t,\lambda)]^{-1}$; c) the ring contains operators with symbols 1, $\theta(t)$, $\Omega(t)$, $\Omega(\lambda)$, where Ω is an arbitrary finite infinitely differentiable function, and θ is the Heaviside function (instead of requiring that operators with symbols $\Omega(t)$ and $\Omega(\lambda)$ be present in the ring, it is sufficient to assume that the ring contains at least ⋯

as well as one completely continuous operator with a nonnegative symbol, whose modulus is bounded below by a positive constant in any bounded domain).

By the ring σ_2 we shall mean the ring of bounded operators of the form (4) satisfying all the conditions of the preceding definition, but with the function $\theta(t)$ in condition c) replaced by the function $\theta(\lambda)$.

An example of the ring σ_1 is furnished by the totality of operators satisfying the conditions of Theorem 4. Under certain additional conditions, the totality of operators satisfying the conditions of Theorem 3 forms the ring σ_2. A simpler example of the ring σ_1 (σ_2) is the totality of operators with symbols of the form

$$a(t) + \sum_{i=1}^{n} b_i(t) K_i(\lambda) \left(c(t) + \sum_{i=1}^{n} d_i(t) \theta(\lambda - c_i) \right),$$

where $a(t)$ and $b_i(t)$ are piecewise constant bounded functions ($c(t)$ and $d_i(t)$ satisfy a Lipschitz condition on the whole axis, and for each of these functions there exists an interval outside of which it is constant), and $K_i(\lambda)$ is the Fourier transform of a summable function.

4. **Theorem 5.** *If the symbol $K(t, \lambda)$ of the operator $A \in \sigma$ ($\sigma = \sigma_{1,2}$), outside some circle $t^2 + \lambda^2 \leq R^2$, satisfies the inequality $|K(t, \lambda)| \geq c > 0$, then the operator A admits a regularization (^{3}) by an operator from the same ring.*

Corollary. If there exist uniform limits

$$\lim_{t \to \pm\infty} K(t, \lambda) = K(\pm\infty, \lambda), \qquad \lim_{\lambda \to \pm\infty} K(t, \lambda) = K(t, \pm\infty)$$

and

$$\min \left[\inf_t |K(t, \pm\infty)|, \ \inf_\lambda |K(\pm\infty, \lambda)| \right] > 0,$$

then the assertion of Theorem 5 remains valid.

Theorem 6. *Under the conditions of the preceding theorem or of its corollary, the known theorems of F. Noether (^{4,5}) hold for the operator A.*

5. **Lemma.** *If the symbol $K(t, \lambda)$ of the operator $A \in \sigma$ satisfies the conditions of the corollary to Theorem 5 and if the closure of the set of values of the functions $K(t, \pm\infty)$ and $K(\pm\infty, \lambda)$ does not intersect a certain ray issuing from the origin, then the index of the operator A is equal to zero.*

The proof of the lemma is similar to the proof of the analogous proposition in (^{6}).

Theorem 7. *Suppose that for the symbol of the operator $A \in \sigma_1$ the following conditions are fulfilled: 1) there exist uniform limits $K(t, \pm\infty)$ and $K(\pm\infty, \lambda)$; the operators with symbols $K(\pm\infty, \lambda)$ belong to the ring σ_1; 2)*

$$\min \left[\inf_\lambda |K(\pm\infty, \lambda)|, \ \inf_t |K(t, \pm\infty)| \right] > 0;$$

3) the closure of the set of values of the functions $K(t, \lambda)/K(-\infty, \lambda)$ for $t < 0$ and $K(t, \lambda)/K(+\infty, \lambda)$ for $t > 0$ does not intersect a certain ray issuing from the origin.

Under these conditions the index of the operator A is equal to the index of the operator

$$A_\infty \varphi = \begin{cases} \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} K(-\infty, \lambda) \Phi(\lambda) \exp(-i\lambda t) \, d\lambda, & \text{for } t < 0, \\ \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} K(+\infty, \lambda) \Phi(\lambda) \exp(-i\lambda t) \, d\lambda, & \text{for } t > 0. \end{cases}$$

Theorem 8. *Suppose that for the symbol of the operator $A \in \sigma_2$ the following conditions are fulfilled: 1) there exist uniform limits $K(t, +\infty)$ and $K(\pm\infty, \lambda)$; the operators with symbols $K(t, \pm\infty)$ belong to the ring σ_2; 2) the same as in Theorem 7; 3) the same as in Theorem 7, but with the roles of the variables t and λ interchanged.*

Then the index of the operator A is equal to the index of the operator

$$A^\infty \varphi = \frac{1}{2}[K(t, +\infty) + K(t, -\infty)] + \frac{1}{2\pi j}[K(t, +\infty) - K(t, -\infty)] \int_{-\infty}^{\infty} \frac{\varphi(\tau)\, d\tau}{t - \tau}.$$

It can be shown that if, in the conditions of Theorem 7 (8), the functions $K(\pm\infty, \lambda)$ $(K(t, \pm\infty))$ are continuous on the closed axis and the quantity

$$\operatorname{Arg} K(+\infty, \lambda)\big|_{-\infty}^{\infty} \quad (\operatorname{Arg} K(t, +\infty))$$

is finite, then

$$\operatorname{ind} A = \frac{1}{2\pi} \operatorname{Arg} \frac{K(-\infty, \lambda)}{K(+\infty, \lambda)}\bigg|_{-\infty}^{\infty}$$

$$\left(\operatorname{ind} A = \frac{1}{2\pi} \operatorname{Arg} \frac{K(t, +\infty)}{K(t, -\infty)}\bigg|_{-\infty}^{\infty}\right).$$

Theorem 9. Let the operator $A \in \sigma_1 \cap \sigma_2$; let: 1) conditions 1) and 2) of Theorems 7 and 8 be satisfied; 2) $A_\infty A \in \sigma_2$ (or $A^\infty A \in \sigma_1$); 3) the limits

$$\lim_{t \to \pm\infty} K(t, \pm\infty) \quad \text{and} \quad \lim_{\lambda \to \pm\infty} K(\pm\infty, \lambda)$$

exist. Under these conditions the index of the operator A is equal to the sum of the indices of the operators A_∞ and A^∞.

If, moreover, the functions $K(t, \pm\infty)$ and $K(\pm\infty, \lambda)$ are continuous on the closed axis and the numbers

$$\operatorname{Arg} K(t, \pm\infty)\big|_{-\infty}^{\infty} \quad \text{and} \quad \operatorname{Arg} K(\pm\infty, \lambda)\big|_{-\infty}^{\infty}$$

are finite, then

$$\operatorname{ind} A = \frac{1}{2\pi} \operatorname{Arg} K(-\infty, \lambda) K(\lambda, +\infty)\, [K(+\infty, \lambda) K(\lambda, -\infty)]^{-1}\bigg|_{-\infty}^{\infty}.$$

In deriving the formulas for the index, results of the papers ([7,8]) were used.

6. Suppose that for the symbols of the operators from the ring σ under consideration there exist uniform limits $K(t, \pm\infty)$ and $K(\pm\infty, \lambda)$, and that the sets of values of the functions $K(t, \pm\infty)$ and $K(\pm\infty, \lambda)$, $-\infty < t, \lambda < \infty$, are closed. In addition, suppose that for complete continuity of an operator from the ring under consideration it is necessary and sufficient that

$$K(t, \pm\infty) = K(\pm\infty, \lambda) = 0.$$

(This condition is satisfied for the rings σ given as examples in item 3. Under fairly broad assumptions about the ring, one can prove the necessity of the condition

$$K(t, \pm\infty) = K(\pm\infty, \lambda) = 0$$

for complete continuity of the corresponding operator.) Under the assumptions made, it can be shown, using methods of the theory of normed rings, that in the case $p = 2$, in order that there exist a bounded regularizer for an operator $A \in \sigma$, it is necessary and sufficient that the conditions

$$K(\pm\infty, \lambda) \neq 0, \qquad K(t, \pm\infty) \neq 0$$

be satisfied. In the case $p \neq 2$, $1 < p < \infty$, this assertion remains valid, at any rate, for operators whose symbols are real or are even (odd) functions of at least one of the variables t or λ.

7. Results analogous to those established hold for systems

$$\sum_{j=1}^{n} \frac{1}{2\pi} \int_{-\infty}^{\infty} K_{lj}(t, \lambda) \Phi_j(\lambda) \exp(-i\lambda t)\, d\lambda = f_l(t), \qquad l = 1, 2, \dots, n.$$

In particular, under certain conditions, Noether theorems hold for such systems, and their index is equal to the index of an operator of the form (4) with symbol

$$\det \| K_{lj}(t, \lambda) \|.$$

In conclusion, the author considers it his duty to express gratitude to Prof. S. G. Mikhlin for suggesting the topic and for guidance.

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Note: Figure translations are in progress. See original paper for figures.

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