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D. B. DOICHINOV

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Abstract

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MATHEMATICS

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ON THE EMBEDDING OF UNIFORM SPACES* IN HILBERT AND EUCLIDEAN SPACES

(Presented by Academician P. S. Aleksandrov on 7 VII 1960)

P. S. Urysohn, in his famous metrization theorem, proved that every metric space with a countable base is homeomorphically embedded in Hilbert space (^{1}). If, moreover, a metric space with a countable base has finite dimension n, then it can be homeomorphically embedded in Euclidean space ($^{2-4}$) of dimension $2n+1$. Yu. M. Smirnov posed the question of finding sufficient conditions for a uniform** (in both directions) embedding of uniform (even just metric spaces with a countable base!) spaces in Hilbert (generalized (^{10}), generally speaking) or Euclidean spaces (^{5}).

Since for compacta every homeomorphism is uniform, and the uniform dimension*** coincides with the ordinary one for them, the two above-mentioned assertions are true for compacta also in the uniform sense. They are also valid in the uniform sense for totally bounded metric spaces (^{5}).

In the present note, leaving open the question of the uniform embedding of arbitrary metric spaces with a countable base in Hilbert space, we give (see Theorem 2) the following condition, stronger than total boundedness, sufficient for a metric space of arbitrary weight to be uniformly embedded in a (generalized) Hilbert space of the same weight (even in its unit ball):

K. *In every uniform covering one can inscribe a finite-multiplicity uniform covering.*

This condition is certainly satisfied for finite-dimensional spaces. The assertion of interest to us follows from the following general assertion:

Theorem 1. *Every uniform space*** X satisfying condition K can be uniformly embedded in the (uniform) product*****

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* A uniform space will be defined by means of a system (structure) of coverings (see (^{8,9})), which we shall call **uniform**. A metric space is considered only in the natural uniform structure of all such coverings, into each of which one can inscribe one of the coverings of the form $\omega_\varepsilon = \{O_\varepsilon x : x \in X\}$.

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** A mapping f is called **uniformly continuous** if the inverse image $f^{-1}\gamma = \{f^{-1}(\Gamma) : \Gamma \in \gamma\}$ of every uniform covering γ of the space Y is a uniform covering for X.

*** **Uniform dimension** (see ([6,7])) is the smallest of the numbers $n = 0, 1, 2, \ldots$ such that in every uniform covering one can inscribe a uniform covering of multiplicity $\leq n + 1$.

**** Only T_1-spaces are considered.

***** The **product structure** of uniform spaces Y_α, by definition, consists of all such coverings into each of which is inscribed a covering of the form $\{\Gamma_{\alpha_1} \times \cdots \times \Gamma_{\alpha_s} \times \prod_{\alpha \neq \alpha_i} Y_\alpha\}$, where $\alpha_1, \ldots, \alpha_s$ is an arbitrarily chosen finite set of indices, Γ_{α_i} is an element of an arbitrarily chosen uniform covering γ_{α_i} of the space Y_{α_i}.

$\prod_\alpha S_\alpha^\tau$ of unit balls S^τ, taken from one and the same generalized Hilbert space H^τ of the same topological weight τ as the space X, whose number is equal to the uniform weight* of the space X.

Finally, in connection with the second question, an example is given here of a uniformly zero-dimensional (and even uniformly locally bicompact**) metric space with a countable base which cannot be uniformly embedded in any Euclidean space.

Plan of the proof of Theorem 1.

Lemma 1. If f_α are mappings*** of the space X into the corresponding spaces Y_α, then the mapping $f = \{f_\alpha\}$, which assigns to each point $x \in X$ the point $y = \{f_\alpha(x)\}$ of the (uniform) product $\prod Y_\alpha$, is uniformly continuous.

Indeed, a base of the uniform structure of the product $\prod Y_\alpha$ consists of all covers of the form

$$\gamma = \{\Gamma_{\alpha_1} \times \ldots \times \Gamma_{\alpha_s} \times \prod_{\alpha \neq \alpha_i} Y_\alpha\},$$

where $\Gamma_{\alpha_i} \in \gamma_{\alpha_i}$, and γ_{α_i} is an arbitrary cover of the space Y_{α_i}. The inverse image

$$f_{\alpha_i}^{-1}(\gamma_{\alpha_i}) = \{f_{\alpha_i}^{-1}(\Gamma) : \Gamma \in \gamma_{\alpha_i}\}$$

of the cover γ_{α_i} under the mapping f_{α_i} is a uniform cover. Since

$$f^{-1}(\Gamma_{\alpha_1} \times \ldots \times \Gamma_{\alpha_s} \times \prod_{\alpha \neq \alpha_i} Y_\alpha) = f_{\alpha_1}^{-1}(\Gamma_{\alpha_1}) \cap \ldots \cap f_{\alpha_s}^{-1}(\Gamma_{\alpha_s}),$$

the inverse image $f^{-1}(\gamma)$ is the "product" **** of a finite number of covers $f_{\alpha_i}^{-1}(\gamma_{\alpha_i})$, and therefore is itself uniform. Hence f is uniformly continuous.

Lemma 2. If f_α are arbitrary mappings of the space X into the corresponding spaces Y_α, and if, moreover, the inverse images $f_\alpha^{-1}(\gamma)$ of covers γ of the spaces Y_α constitute a pseudobase***** of the uniform structure of the space X, then the mapping $f = \{f_\alpha\}$ is one-to-one and the inverse mapping f^{-1} is uniformly continuous.

Indeed, for distinct points x and y there exists such a product of covers $f_{\alpha_i}^{-1}(\gamma_{\alpha_i})$, $i = 1, \dots, s$, that $x \in U$, but $y \notin U$, for some set

$$U = f_{\alpha_1}^{-1}(\Gamma_{\alpha_1}) \cap \dots \cap f_{\alpha_s}^{-1}(\Gamma_{\alpha_s}),$$

where $\Gamma_{\alpha_i} \in \gamma_{\alpha_i}$. Then $f_{\alpha_i}(x) \in \Gamma_{\alpha_i}$, but $f_{\alpha_i}(y) \notin \Gamma_{\alpha_i}$ for some α_i. Hence $f(x) \neq f(y)$. Next, let $\omega = \{U_\lambda\}$ be a cover of the space X. Take covers $f_{\alpha_i}^{-1}(\gamma_{\alpha_i})$, $i = 1, \dots, s$, whose product is inscribed in ω. If

$$f_{\alpha_1}^{-1}(\Gamma_{\alpha_1}) \cap \dots \cap f_{\alpha_s}^{-1}(\Gamma_{\alpha_s}) \subset U_\lambda,$$

then

$$\Gamma_{\alpha_1} \times \dots \times \Gamma_{\alpha_s} \times \prod_{\alpha \neq \alpha_i} Y_\alpha \subset f(U_\lambda).$$

Thus the cover

$$\{\Gamma_{\alpha_1} \times \dots \times \Gamma_{\alpha_s} \times \prod_{\alpha \neq \alpha_i} Y_\alpha\}$$

is inscribed in $f(\omega)$. Therefore $f(\omega)$ is a uniform cover. In view of the arbitrary choice of the cover ω, the mapping f^{-1} is uniformly continuous.

Lemma 3. If $\omega = \{U_\lambda\}$ is a finite-fold cover of cardinality τ, then there exists a mapping f of the space X into the ball S^τ of a generalized Hilbert space of weight τ such that the preimage of some space $Y = f(X)$ is inscribed in ω.

Indeed, there exists a uniformly continuous bounded pseudometric d on X for which the given cover ω of multiplicity n is uniform (^{9}). Let, for example,

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[*] A **uniform base** is any system of uniform covers such that into every uniform cover one can inscribe a cover of this system. The **uniform weight** of a uniform space is the least of the cardinalities of all its uniform bases.

[**] A uniform space is called (7) **uniformly locally bicompact** if there exists a uniform cover consisting of closed bicompact sets.

[***] Everywhere below, unless explicitly stated otherwise, mappings are assumed to be uniformly continuous, spaces and covers—uniform.

[****] The "product" of covers γ_i is the cover whose elements are all possible intersections $\bigcap \Gamma_i$, where $\Gamma_i \in \gamma_i$.

[*****] A **pseudobase** of a uniform space is any system of uniform covers such that into every uniform cover one can inscribe the product of a finite number of covers of this system.

$d(x, y) \leqslant 1/\sqrt{n}$ for any points x, y of X. There exists a cover $\omega_\varepsilon = \{O_\varepsilon x\}$ (ε-neighborhoods are taken in the pseudometric d) inscribed in the cover ω. Put

$$f_\lambda(x) = d(x, X \setminus U_\lambda).$$

For any points x and y of X we have: 1) $f_\lambda(x) \leqslant 1/\sqrt{n}$; 2) no more than n of the numbers $f_\lambda(x)$ are nonzero, and at least one of them is $\geqslant \varepsilon$; 3) $|f_\lambda(x) - f_\lambda(y)| \leqslant d(x, y)$.

The mapping $f = \{f_\lambda\}$ uniformly maps X into S^τ, since $\rho(f(x), 0) \leqslant 1$ and $\rho(f(x), f(y)) \leqslant \sqrt{2n}\, d(x, y)$. By virtue of 2), the sets

$$\Gamma_\lambda = \mathcal{E}\{f(x) : f_\lambda(x) > 0\}$$

form a uniform cover γ of the image $Y = f(X)$, and the preimage $f^{-1}(\gamma)$ is inscribed in ω.

Let us indicate the derivation of Theorem 1. Take a base $\{\omega_\alpha\}$ consisting of finite-multiplicity covers, whose cardinality is equal to the uniform weight. We may assume that the cardinality of each cover ω_α is equal to the topological weight τ of the space X. By Lemma 3, for each α there exists a mapping f_α of the space X into the ball S^τ and a cover γ_α of the image $f_\alpha(X)$ such that

$$f_\alpha^{-1}(\gamma_\alpha)$$

is inscribed in ω_α. The system of covers $\{f_\alpha^{-1}(\gamma_\alpha)\}$ turns out to be a base. By Lemmas 1 and 2, the space X is uniformly homeomorphic to the image $f(X)$ lying in $\prod S^\tau$, where $f = \{f_\alpha\}$.

Let now a countable number of uniformly metrizable spaces Y_i be given. It is known that the product $\prod Y_i$ can be metrized in the following way: if ρ_i is a metric of the space Y_i (one may assume (^{5}) that the diameter of the space Y_i in the metric ρ_i is not greater than 1), then for any points $y' = \{y_i'\}$ and $y'' = \{y_i''\}$ of the product $\prod Y_i$ we put

$$\rho(y', y'') = \left[\sum_i 2^{-i} \rho_i^2(y_i', y_i'')\right]^{1/2}.$$

We shall agree to speak of the **metric product** of the spaces Y_i, meaning precisely this metric ρ.

Lemma 4. *For a countable number of uniformly metrizable spaces Y_i, their uniform and metric products are uniformly homeomorphic.*

Indeed, let γ be a cover of the uniform product. Inscribe in it a cover π_k of the form

$$\{\Gamma_1 \times \cdots \times \Gamma_k \times \prod_{i>k} Y_i\},$$

where $\Gamma_i \in \gamma_i$, and γ_i is a cover of the space Y_i $(i \leqslant k)$. There exists $\varepsilon > 0$ such that the cover ω^i_ε of the space Y_i (consisting of the ε-neighborhoods of all its points) is inscribed in γ_i. For the number

$$\delta = \varepsilon/\sqrt{2^k},$$

the cover $\omega_\delta = \{O_\delta y\}$ of the metric product is inscribed in the cover π_k, and hence also in γ. Conversely, for an arbitrary number $\delta > 0$ and a natural number N, chosen according to δ so that

$$\frac{1}{2}^{N} < \delta^2/2,$$

the cover $\omega_{\delta,N}$ of the form

$$\{O_r y_1 \times \cdots \times O_r y_N \times \prod_{i>N} Y_i\},$$

where

$$r = \delta/\sqrt{N},$$

and $O_r y_i$ is the neighborhood of the point y_i in Y_i (y_i runs through the entire set of points of the space Y_i), is inscribed in the cover $\omega_\delta = \{O_\delta y\}$ of the metric product.

Theorem 2. *Every metric space of topological weight τ satisfying condition K can be uniformly embedded in the unit ball of a generalized Hilbert space of weight τ.*

Let us indicate the proof. Take a base $\{\omega_i\}$ of the space Y, consisting of a countable number of finite-multiplicity covers of cardinality $\leqslant \tau$. By Theorem 1, the space Y is uniformly embedded in the uniform product

$$\prod_{i=1}^{\infty} S^\tau_i,$$

and this, by Lemma 4, is uniformly embedded in the ball S^τ.

Corollary. *Every metric space of finite uniform dimension can be uniformly embedded in the unit ball of a generalized Hilbert space of the same weight.*

Example. Consider an abstract set R, consisting of a countable number of points x^i_{nk}, where $n = 1, 2, \ldots;\ k = 1, 2, \ldots;\ i = 1, 2, \ldots, n$. Introduce in it a distance ρ: let $\rho(x^i_{nk}, x^j_{mr}) = 1$, if $n \neq m$ or $k \neq r$; $\rho(x^i_{nk}, x^j_{nk}) = 1/k$, if $i \neq j$, and, finally, $\rho(x^i_{nk}, x^i_{nk}) = 0$. Since in every ball of radius $1/2$ there is only a finite number of points, R is uniformly locally bicompact. Let $\varepsilon > 0$, and let $\delta = 1/r < \varepsilon$. The system of δ-neighborhoods of all points x^i_{nk} for which $k \leq r-1$, and of ε-neighborhoods of all points x^1_{nk} for which $k \geq r$, is a uniform discrete cover (since $O_\varepsilon x^i_{nk} = O_\varepsilon x^1_{nk}$ for $k \geq r$) inscribed in ω_ε. Thus, R is uniformly zero-dimensional.

Call a metric space ε-**discrete** if all positive distances in it are $\geq \varepsilon$. If the "coordinates" n and k of the points $x^i_{nk} \in R$ are fixed, then we obtain a $1/k$-discrete set D_{nk}. Let $\delta(\varepsilon)$ be the "modulus of continuity"of a one-to-one mapping f'. Under this assumption, the complete preimage of an ε-discrete set will be $\delta(\varepsilon)$-discrete. Suppose, finally, that there exists a uniformly homeomorphic mapping f of the space R into a Euclidean space E^N. Let d be the usual metric in E^N. By uniform continuity there exists a number K such that, whenever $\rho(x, x') \leq 1/K$, necessarily $d(f(x), f(x')) < 1$. Hence, $d(f(x^i_{nK}), f(x^1_{nK})) < 1$ for all n and i, where $i \leq n$. Let $\delta(\varepsilon)$ be the modulus of continuity of the inverse mapping f^{-1}, and let $\eta = \delta(1/K)$. The image of the set D_{nK} (for any n) is η-discrete and lies in the unit ball S^N of the Euclidean space E^N, which is impossible, since the number of points of any η-discrete set of the ball S^N does not exceed the number

$$\left(\frac{2+\eta}{\eta}\right)^N,$$

whereas the number of points in D_{nK} grows with n.

Remark. No bounded set of the space R other than a singleton (being completely bounded) is uniformly separated (^{5}). Therefore E. Gorin’ s arguments (^{5}) do not apply here.

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REFERENCES

[1] P. S. Urysohn, *Math. Ann.*, **94**, 309 (1925).
[2] K. Menger, *Proc. Acad. Wetensch. Amst.*, **29**, 476 (1926).
[3] Nöbeling, *Math. Ann.*, **104**, 71 (1930).
[4] W. Hurewicz, *Proc. Acad. Wetensch. Amst.*, **30**, 425 (1927).
[5] E. Gorin, *UMN*, **14**, No. 5, 129 (1959).
[6] Yu. M. Smirnov, *Matem. sborn.*, **38**, No. 3, 283 (1956).
[7] J. Isbell, *Pacif. J. Math.*, **9**, 107 (1959).
[8] Yu. M. Smirnov, *Matem. sborn.*, **31**, No. 3, 543 (1952).
[9] J. Kelley, *General Topology*, N. Y., 1955.
[10] Yu. M. Smirnov, *UMN*, **6**, No. 6, 100 (1951).

Note: Figure translations are in progress. See original paper for figures.

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