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Abstract

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B. V. SHABAT

THE METHOD OF MODULI IN SPACE

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In the geometric theory of functions of a complex variable, the notion of the modulus of topological quadrilaterals and rings is widely used. In the last decade this notion has received a new interpretation in the so-called method of extremal lengths, originating with the memoir of Ahlfors and Beurling ([1]). This interpretation makes it possible to transfer the method of moduli to Euclidean spaces of any number of dimensions; for greater clarity we shall restrict ourselves to the case of three-dimensional space.

1°. **Definition 1. Family of lines.** Let in a spatial domain D there be given some family $\{C\}$ of rectifiable curves, and also a **metric**—a measurable nonnegative function $\rho = \rho(P)$, where P is a point of D. We shall call the metric ρ **admissible** if, for every curve C of the family,

$$\int_C \rho\, ds \geqslant 1.$$

The lower bound, over all admissible ρ,

$$M = \inf_{\rho} \int_D \rho^3\, d\omega$$

will be called the **modulus** of the family $\{C\}$. A homeomorphic mapping $P_* = f(P)$ of the domain D onto D_* will be called Q-**quasiconformal** if it has continuous partial derivatives in D, its Jacobian is everywhere positive, and at each point $P \in D$ its principal linear part transforms a sphere into an ellipsoid whose ratio

$$p = \frac{a}{c}$$

of the major semiaxis to the minor one is bounded: $p(P) \leqslant Q$.

Theorem 1. *Under Q-quasiconformal mappings the modulus of a family of curves may change only within bounded limits depending solely on Q; more precisely, if $\{C_*\}$ is the image of the family $\{C\}$, then*

$$M\{C\}/Q^2 \leqslant M\{C_*\} \leqslant Q^2 M\{C\}. \tag{1}$$

Indeed, let $\rho(P)$ be an arbitrary metric admissible for the family $\{C\}$. On D_* define the function

$$\rho_*(P_*) = \rho(P)/\mu(P),$$

where $P = f^{-1}(P_*)$ and $\mu = r/a$ is the minimal stretching of f at the point P (r is the radius of the sphere corresponding to the ellipsoid); ρ_* is, evidently, admissible for $\{C_*\}$, since $ds_* \geqslant \mu\, ds$ (ds_* is the element of length of C_*) and

$$\int_{C_*} \rho_*\, ds_* \geqslant \int_C \rho\, ds \geqslant 1.$$

But the Jacobian of the mapping

$$J = \frac{r^3}{abc} = \frac{\mu^3 a^2}{bc} \leqslant \mu^3 Q^2$$

($a \geqslant b \geqslant c$ are the semiaxes of the ellipsoid transformed into a sphere), therefore

$$M\{C_*\} \leqslant \inf_{\rho_*} \int_{D_*} \rho_*^3\, d\omega_* = \inf_{\rho} \int_D \rho^3 \frac{J}{\mu^3}\, d\omega \leqslant Q^2 M\{C\}.$$

Applying the same reasoning to the mapping f^{-1}, also Q-quasiconformal, we obtain the left-hand side of (1).

Definition 2. Family of surfaces. Let $\{S\}$ be a family of quadrable surfaces in D; the metric $\rho = \rho(P)$ will be con-

be admissible if for any surface S of the family

$$\int_S \rho^2\, d\sigma \geqslant 1.$$

The **modulus** of the family $\{S\}$ will be called

$$M\{S\} = \inf_{\rho} \int_D \rho^3\, d\omega,$$

where the lower bound is taken over all admissible ρ.

Theorem 2. *If* $\{S_*\}$ *is the family corresponding to* $\{S\}$ *under a* Q*-quasiconformal mapping, then*

$$M\{S\}/Q \leqslant M\{S_*\} \leqslant QM\{S\}. \tag{2}$$

From Theorems 1 and 2 it follows that the moduli of families of curves and surfaces are invariant under conformal mappings.

Remark. Using the results of A. I. Markushevich (^{2}), one can substantially weaken the function-theoretic restrictions imposed above in the definition of Q-quasiconformal mappings.

2°. Example 1. D is a rectangular parallelepiped:

$$0 < x < \alpha, \qquad 0 < y < \beta, \qquad 0 < z < \gamma;$$

$\{C\}$ is the family of all rectifiable curves joining the upper and lower bases. We have

$$M\{C\} = \alpha\beta/\gamma^2. \tag{3}$$

Indeed, since the length of any C is not less than γ, $\rho_0 = 1/\gamma$ is an admissible metric, to which there corresponds the volume $M = \alpha\beta/\gamma^2$. Let ρ be an arbitrary admissible metric; by Fubini' s theorem and Hölder' s inequality,

$$\int_D \rho^3\,d\omega = \int_0^\alpha dx \int_0^\beta dy \int_0^\gamma \rho^3\,dz \geqslant \frac{1}{\gamma^2}\int_0^\alpha dx \int_0^\beta dy \left(\int_0^\gamma \rho\,dz\right)^3 \geqslant \frac{\alpha\beta}{\gamma^2},$$

for, since the rectilinear segments belong to $\{C\}$, we have

$$\int_0^\gamma \rho\,dz \geqslant 1;$$

thus (3) is proved.

Example $1'$**.** D is as in Example 1; $\{S\}$ is the family of all quadrable surfaces whose boundaries lie on the lateral faces of D. The metric

$$\rho_0 = 1/\sqrt{\alpha\beta}$$

is admissible, and the corresponding volume is

$$M = \gamma/\sqrt{\alpha\beta}.$$

Let ρ be an arbitrary admissible metric; by Fubini' s theorem and Hölder' s inequality,

$$\int_D \rho^3\, d\omega = \int_0^\gamma dz \iint_{S_z} \rho^3\, d\sigma \geqslant \frac{1}{\sqrt{\alpha\beta}} \int_0^\gamma \left(\iint_{S_z} \rho^2\, d\sigma \right)^{3/2} dz,$$

where S_z is the section of D at height z. Since $S_z \in \{S\}$, we have

$$\iint_{S_z} \rho^2\, d\sigma \geqslant 1$$

and

$$\int_D \rho^3\, d\omega \geqslant \frac{\gamma}{\sqrt{\alpha\beta}}.$$

Thus:

$$M\{S\} = \gamma/\sqrt{\alpha\beta}. \tag{3'}$$

Remark. Let $\{C_1\}$, $\{C_2\}$, and $\{C_3\}$ be the families of rectifiable curves joining the corresponding opposite faces of D. From (3) we have

$$M\{C_1\}M\{C_2\}M\{C_3\} = 1.$$

If $\{S_k\}$ is the family of quadrable surfaces ending on the faces free from the ends of $\{C_k\}$, then from (3) and (3′) we obtain

$$M\{C_k\} = 1/M^2\{S_k\}.$$

Example 2. D is a spherical sector:

$$r_1 < r < r_2, \qquad 0 \leqslant \varphi < 2\pi, \qquad 0 < \theta < \theta_0$$

$(r, \varphi, \theta$ are polar coordinates); $\{C\}$ is the family of all rectifiable cur-

curves connecting the spherical boundaries of D. We have

$$M\{C\} = 4\pi \sin^2(\theta_0/2)/\ln^2(r_2/r_1). \tag{4}$$

Indeed, the metric $\rho_0 = 1/r\ln(r_2/r_1)$ is admissible, since

$$\int_C \rho_0\,ds \geq \int_{r_1}^{r_2} \rho_0\,dr = 1$$

for any C, and the corresponding volume is equal to the right-hand side of (4). Let ρ be an arbitrary admissible metric,

$$\int_D \rho^3\,d\omega = \int_0^{2\pi} d\varphi \int_0^{\theta_0} \sin\theta\,d\theta \int_{r_1}^{r_2} \rho^3 r^2\,dr;$$

but since, if

$$\int_{r_1}^{r_2} \rho\,dr \geq 1,$$

then

$$\int_{r_1}^{r_2} \rho^3 r^2\,dr \geq \frac{1}{\ln^2(r_2/r_1)}$$

(this is proved by the methods of the calculus of variations), it follows that

$$\int_D \rho^3\,d\omega \geq \frac{1}{\ln^2(r_2/r_1)} \int_0^{2\pi} d\varphi \int_0^{\theta_0} \sin\theta\,d\theta = \frac{4\pi\sin^2(\theta_0/2)}{\ln^2(r_2/r_1)},$$

as required.

Example 2′. D as in Example 2; $\{S\}$ is the family of all quadrable surfaces whose boundaries lie on the conical part of the boundary of D. We have

$$M\{S\} = \ln(r_2/r_1)/2\sqrt{\pi}\sin(\theta_0/2). \tag{4'}$$

This is proved by methods analogous to the preceding ones, using Hölder' s inequality.

Remark. From (4) and (4′) it follows that $M\{C\} = 1/M^2\{S\}$.

Example 3. D is a cylindrical ring: $r_1 < r < r_2,\ 0 \leq \varphi < 2\pi,\ 0 < z < H$ (r, φ, z are cylindrical coordinates); $\{C\}$ is the family of all rectifiable curves connecting the upper and lower bases of D; $\{S\}$ is the family of all quadrable surfaces homotopic in D to the rings $r_1 < r < r_2,\ z = z_0$ $(0 < z_0 < H)$. We have

$$M\{C\} = 1/M^2\{S\} = \pi(r_2^2 - r_1^2)/H^2. \tag{5}$$

Example 4. D as in Example 3; $\{C\}$ is the family of all closed rectifiable curves homotopic to the circles $r = r_0,\ z = z_0$ $(r_1 < r_0 < r_2,\ 0 < z_0 < H)$; $\{S\}$ is the family of all quadrable surfaces homotopic to the rectangles $\varphi = \varphi_0,\ r_1 < r < r_2,\ 0 < z < H$. We have

$$M\{C\} = 1/M^2\{S\} = H(r_2 - r_1)/4\pi^2 r_1 r_2. \tag{6}$$

Example 5. D as in Example 3; $\{C\}$ is the family of all rectifiable curves homotopic to the segments $\varphi = \varphi_0,\ z = z_0,\ r_1 < r < r_2$ $(0 < z_0 < H)$; $\{S\}$ is the family of all quadrable surfaces homotopic to the cylinders $r = r_0,\ 0 < z < H$ $(r_1 < r_0 < r_2)$. We have

$$M\{C\} = 1/M^2\{S\} = \pi H/2(\sqrt{r_2} - \sqrt{r_1})^2. \tag{7}$$

3°. General properties. We shall call an admissible metric **extremal** if, in it, the volume of the domain coincides with the modulus of the family of curves or surfaces under consideration. The following theorem asserts the uniqueness of the extremal metric for a given family.

Theorem 3. *The extremal metric of a given family of curves or surfaces is determined uniquely, up to a set of volume measure* 0.

We shall prove the theorem for the case of a family of curves $\{C\}$. Let ρ_1 and ρ_2 be two extremal metrics for this family, i.e.

$$\int_D \rho_1^3\, d\omega = \int_D \rho_2^3\, d\omega = M\{C\}.$$

The metric $\rho = (\rho_1 + \rho_2)/2$ is obviously admissible for this family and, integrating over D the identity

$$\frac{1}{2}(\rho_1^3 + \rho_2^3) = \frac{1}{8}(\rho_1 + \rho_2)^3 + \frac{3}{8}(\rho_1 - \rho_2)^2(\rho_1 + \rho_2),$$

we find

$$\int_D (\rho_1 - \rho_2)^2(\rho_1 + \rho_2)\, d\omega \leq 0,$$

whence $\rho_1 = \rho_2$ almost everywhere in D. For the case of a family of surfaces the proof proceeds similarly, if one chooses the admissible metric

$$\rho = \sqrt{\frac{1}{2}(\rho_1^2 + \rho_2^2)}$$

and uses the identity

$$\frac{1}{2}(\rho_1^3 + \rho_2^3) = \rho^3 + \frac{1}{4}(\rho_1 - \rho_2)^2 \frac{\rho_1^4 + 4\rho_1\rho_2\rho^2 + \rho_2^4}{\rho_1^3 + 2\rho^3 + \rho_2^3}.$$

The following two theorems extend the well-known Grötzsch principles from the theory of conformal mappings (see, for example, (3)).

Theorem 4. *If two families of curves (or two families of surfaces) with moduli* M_1, M_2 *lie in disjoint domains* D_1, D_2*, and a third family, lying in* $D = D_1 \cup D_2$*, is obtained by the union of the given two, then the modulus of the latter satisfies*

$$M \geq M_1 + M_2. \tag{8}$$

Let ρ be an admissible metric for the third family; then $\rho_k = \rho$ in D, $\rho_k = 0$ in $D \setminus D_k$, will obviously be admissible for the k-th family ($k = 1, 2$), and (8) follows from the identity

$$\int_D \rho^3 \, d\omega = \int_{D_1} \rho_1^3 \, d\omega + \int_{D_2} \rho_2^3 \, d\omega.$$

Corollary. Under an "extension" of a family (of surfaces or curves), its modulus can only increase.

Theorem 5. *If the families of curves* $\{C_1\}, \{C_2\}$ *are situated in disjoint domains* D_1, D_2*, and each curve of the family* $\{C\}$ *consists of one* C_1 *and one* C_2*, then*

$$\frac{1}{\sqrt{M\{C\}}} \geq \frac{1}{\sqrt{M\{C_1\}}} + \frac{1}{\sqrt{M\{C_2\}}}. \tag{9}$$

For families of surfaces $\{S_1\}, \{S_2\}$ and $\{S\}$ having the same properties,

$$\frac{1}{M^2\{S\}} \geq \frac{1}{M^2\{S_1\}} + \frac{1}{M^2\{S_2\}}. \tag{10}$$

Let ρ_k be metrics admissible for the families of curves $\{C_k\}$; then for any λ, $0 < \lambda < 1$, the metric $\rho = \lambda \rho_1$ in D_1, $\rho = (1 - \lambda)\rho_2$ in D_2, is obviously admissible for $\{C\}$, and by integration we obtain

$$M\{C\} \leq \lambda^3 M\{C_1\} + (1 - \lambda)^3 M\{C_2\}.$$

Choosing

$$\lambda = \frac{\sqrt{M\{C_2\}}}{\sqrt{M\{C_1\}} + \sqrt{M\{C_2\}}},$$

we obtain (9). For families of surfaces the proof is analogous; one need only put

$$\rho = \sqrt{\lambda}\rho_1 \quad \text{in } D_1, \qquad \rho = \sqrt{1 - \lambda}\rho_2 \quad \text{in } D_2,$$

and then choose

$$\lambda = \frac{M^2\{S_2\}}{M^2\{S_1\} + M^2\{S_2\}}.$$

Corollary. Under a “lengthening”of a family (of curves or surfaces), its modulus can only decrease.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.

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