ON THE CONCEPT OF A GENERALIZED SOLUTION

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Abstract

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ON THE CONCEPT OF A GENERALIZED SOLUTION

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One of the important problems of the modern theory of differential equations consists in clarifying the concept of a generalized solution of quasilinear hyperbolic systems. Here we are concerned with so-called "divergent" systems

$$\frac{\partial F_i(q_1, q_2, \dots, q_n)}{\partial t} + \frac{\partial G_i(q_1, q_2, \dots, q_n)}{\partial x} = 0. \tag{1}$$

Generalized solutions are functions $q_1(x,t), q_2(x,t), \dots, q_n(x,t)$ for which, along any contour,

$$\oint F_i\,dx - G_i\,dt = 0.$$

However, not all functions satisfying this condition should be regarded as generalized solutions.

In the paper ([1]), I. M. Gelfand proposed calling generalized solutions those $\{q_1, q_2, \dots, q_n\}$ which can be obtained as limits as $\varepsilon \to +0$ of solutions of the following system with "viscosity" :

$$\frac{\partial F_i}{\partial t} + \frac{\partial G_i}{\partial x} = \frac{\partial}{\partial x}\left(\varepsilon b_{ik}\frac{\partial q_k}{\partial x}\right). \tag{2}$$

The matrix $\|b_{ik}\|$ must ensure the evolutionary character of system (2). It is clear that the concept of a generalized solution of system (1), defined in this way, is meaningful only in the case where it is determined uniquely by system (1) and does not depend on the particular form of the matrix $\|b_{ik}\|$.

I. M. Gelfand posed the question: is the hyperbolicity of system (1) sufficient for the concept of a generalized solution to be meaningful? We shall give a negative answer to this question.

Let us consider an example of a system of two equations of the following special form:

$$\frac{\partial L_{q_1}}{\partial t} + \frac{\partial L^1_{q_1}}{\partial x} = \frac{\partial}{\partial x}(\varepsilon D_{s_1}),$$

$$\frac{\partial L_{q_2}}{\partial t} + \frac{\partial L^2_{q_2}}{\partial x} = \frac{\partial}{\partial x}(\varepsilon D_{s_2}), \tag{3}$$

$$L = L(q_1, q_2), \qquad L^1 = L^1(q_1, q_2), \qquad s_1 = \frac{\partial q_1}{\partial x}, \qquad s_2 = \frac{\partial q_2}{\partial x},$$

$$D = b_{11}s_1^2 + 2b_{12}s_1s_2 + b_{22}s_2^2, \qquad b_{ik} = b_{ik}(q_1, q_2).$$

It can be verified that the convexity of $L(q_1, q_2)$ ensures the hyperbolicity of system (3) for $\varepsilon = 0$. (For strict hyperbolicity it is necessary

require in addition the absence of multiple roots of det $\|L_{q_iq^k} + \lambda L^1_{q_iq^k}\|$. The evolvability (in the linear approximation) of system (3) is ensured by the positive definiteness of $\|b_{ik}\|$.

Consider solutions of system (3) of the form

$$q_1 = q_1(\tau), \qquad q_2 = q_2(\tau), \qquad \tau = \frac{t - \alpha x}{\varepsilon}$$

for some fixed α. To find q_1, q_2 we obtain the ordinary equations

$$\frac{1}{\varepsilon}(L_{q_i})' - \frac{\alpha}{\varepsilon}(L^1_{q_i})' = -\alpha(D_{s_i})', \qquad s_i = -\frac{\alpha}{\varepsilon}q_i'.$$

If we denote

$$\sigma_i = \frac{\varepsilon}{\alpha}s_i, \qquad D(s_1, s_2) = \frac{\alpha^2}{\varepsilon^2}D(\sigma_1, \sigma_2), \qquad D_{s_i}(s_1, s_2) = \frac{\alpha}{\varepsilon}D_{\sigma_i}(\sigma_1, \sigma_2),$$

then these equations take a form independent of the parameter ε:

$$(L_{q_i} - \alpha L^1_{q_i} + \alpha^2 D_{\sigma_i})' = 0, \qquad \frac{dq_i}{d\tau} = -\sigma_i. \tag{4}$$

We shall seek solutions of system (4) that tend, as $\tau \to \pm\infty$, to finite limits in such a way that $dq_i/d\tau \to 0$. Such solutions, as $\varepsilon \to 0$, become discontinuities propagating with velocity $\dfrac{dx}{dt} = \dfrac{1}{\alpha}$ (see (1)). We shall show that the set of discontinuous generalized solutions of system (1) obtained in this way may depend on the particular form of the matrix $\|b_{ik}\|$.

System (4) has the integrals

$$L_{q_1} - \alpha L^1_{q_1} + \alpha^2 D_{\sigma_1} = A_1 = \text{const},$$

$$L_{q_2} - \alpha L^1_{q_2} + \alpha^2 D_{\sigma_2} = A_2 = \text{const}. \quad (5)$$

Since as $\tau \to \pm\infty$, $D_{\sigma_i} \to 0$, we arrive at the conditions $[L_{q_i}] - \alpha[L^1_{q_i}] = 0$, where the symbol $[f]$ denotes $f(+\infty) - f(-\infty)$. As $\varepsilon \to 0$, $[f]$ becomes the difference between the values of f on the two sides of the discontinuity. It is clear that the limiting values of q_i as $\tau \to \pm\infty$ determine stationary points of the function Λ:

$$\Lambda = L - \alpha L^1 - A_1 q_1 - A_2 q_2.$$

Using the function Λ, the equations for q_i can be written in the following elegant form:

$$\Lambda_{q_1} = -\alpha^2 D_{\sigma_1}, \qquad dq_1 = -\sigma_1 d\tau;$$

$$\Lambda_{q_2} = -\alpha^2 D_{\sigma_2}, \qquad dq_2 = -\sigma_2 d\tau. \quad (6)$$

The left column of equalities (6) shows that the vector (σ_1, σ_2) is orthogonal, in the sense of the metric defined by the form D, to the level line $\Lambda = \text{const}$. Indeed, a vector (r_1, r_2) tangent to the level line satisfies the equality

$$r_1 \Lambda_{q_1} + r_2 \Lambda_{q_2} = 0,$$

from which it follows that $r_1 D_{\sigma_1} + r_2 D_{\sigma_2} = 0$, i.e. that

$$2[b_{11}\sigma_1 r_1 + b_{12}(\sigma_1 r_2 + \sigma_2 r_1) + b_{22}\sigma_2 r_2] = 0.$$

Thus, equations (6) determine motion along the line of "steepest descent" on the surface of the graph $\Lambda = \Lambda(q_1, q_2)$ in the direction of increasing Λ as τ increases. Indeed,

$$d\Lambda = \Lambda_{q_1} dq_1 + \Lambda_{q_2} dq_2 = \alpha^2(\sigma_1 D_{\sigma_1} + \sigma_2 D_{\sigma_2})\, d\tau = 2\alpha^2 D\, d\tau.$$

After the trajectory in the q_1, q_2 plane has been determined, τ along it can be found by quadrature

$$\tau = -\int \frac{dq_1}{\sigma_1} = -\int \frac{dq_2}{\sigma_2}.$$

Fig. 1

Figure 1: Fig. 1

The integral diverges as one approaches a nondegenerate stationary point of the function Λ.

The trajectories of interest to us connect two arbitrary stationary points of Λ. From the divergence of the integral it follows that along such trajectories τ varies from $-\infty$ to $+\infty$.

Let $\Lambda(q_1, q_2)$ have 5 stationary points I, II, III, IV, V, where I, III, V are minima, and II and IV are minimaxes, and let the level lines of Λ have the form shown in Fig. 1. The trajectories that may be taken as admissible here are those connecting the minimaxes II and IV with one of the minima I, III, V. From each point II, IV two trajectories go out in the direction of smaller values. One of the trajectories issuing from point II falls into minimum I, and the other into minimum III. From point IV, one trajectory goes to minimum V, while the other may, depending on the metric D, fall into I or into III. Both cases are possible, as is easily verified by drawing from IV a trajectory that descends all the time along the graph Λ to one of the minima, and then constructing $\|b_{ij}(q_1, q_2)\|$ so that, in this metric, the displacements along the trajectory are orthogonal to the level lines. The construction of $\|b_{ij}\|$ is elementary.

Fig. 1

Thus, depending on the viscosity matrix, system (3) for $\varepsilon = 0$ may admit, as generalized solutions, different sets of discontinuities. For this it is sufficient that, for certain α, A_1, A_2, the level lines of the function have the form of Fig. 1. For one viscosity the admissible discontinuities will correspond to the transitions $I \to II, III \to II, I \to IV, V \to IV$, and for another—to the transitions $I \to II, III \to II, III \to IV, V \to IV$.

It remains for us to give an example of functions L and L^1 and constants α, A_1, A_2 that lead to the described picture of level lines for Λ. They are:

$$L = \frac{q_1^2}{2} + \frac{10q_2^6 - 108q_2^5 + 255q_2^4 + 180q_2^3 + 146460q_2^2}{294000},$$

$$L^1 = q_1 q_2, \qquad \alpha = 1, \qquad A_1 = A_2 = 0.$$

List of stationary points of Λ:

$$
\begin{aligned}
I.&\quad q_1 = 1, \quad & q_2 = -1.\\
II.&\quad q_1 = 0, \quad & q_2 = 0.\\
III.&\quad q_1 = -1, \quad & q_2 = 1.\\
IV.&\quad q_1 = -3, \quad & q_2 = 3.\\
V.&\quad q_1 = -6, \quad & q_2 = 6.
\end{aligned}
$$

Let us also note that if one sets $q_1 = u,\ L_{q_2} = v,\ q_2 = \varphi(v)$, then our example takes the form of the well-studied system

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(v)}{\partial x} = 0,$$

$$\frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} = 0.$$

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REFERENCES CITED

1. I. M. Gel' fand, *Uspekhi Mat. Nauk*, **14**, no. 2 (86), 87 (1959).

Note: Figure translations are in progress. See original paper for figures.