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Abstract

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ON DENSELY EMBEDDED IDEALS OF SEMIGROUPS

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In the theory of semigroups of transformations, the concept of a densely embedded ideal has recently begun to play a noticeable role.

An ideal N of a semigroup T is called **densely embedded** (^{1,2}) if the following two conditions hold: 1) every homomorphism of the semigroup T that is not an isomorphism (we shall say: a proper homomorphism) induces in N also a proper homomorphism; 2) for every semigroup S containing T and different from T, such that N is an ideal in S, there exists a proper homomorphism of the semigroup S inducing an isomorphism in N.

With the aid of densely embedded ideals one can obtain an abstract characterization of certain very important semigroups of transformations (see, for example, ($^{2-4}$)). Often the presence in a semigroup of a densely embedded ideal with one or another property characterizes the semigroup uniquely up to isomorphism. In this connection the following question is of interest: under what conditions can a semigroup N be a densely embedded ideal of some semigroup*.

In the present note a necessary condition of this kind is given. Besides the main result, the construction used here for extending a semigroup is, in our opinion, of some interest.

We first give the definitions we shall need.

The set of all elements x of a semigroup T satisfying the condition $xT = Tx = 0$, where 0 is the zero of the semigroup T, is called the **annihilator** of T. We shall denote the annihilator of the semigroup T by $A(T)$. A nonempty annihilator will be called **nontrivial** if it contains elements different from zero. An ideal N of a semigroup T is called **characteristic** if it is an ideal in every semigroup T^* containing T as an ideal.

The following lemma is easily proved.

Lemma 1. *$A(T)$, if it is nonempty, is a characteristic ideal of the semigroup T.*

We now formulate the main result of the note.

Theorem. *A semigroup possessing a nontrivial annihilator cannot be a densely embedded ideal in any semigroup.*

Let N be a semigroup with nontrivial annihilator, and let a be an arbitrary nonzero element of $A(N)$, $a \in A(N)$, $a \neq 0$. Suppose N is an ideal in a semigroup T, and that every proper homomorphism of T induces a proper homomorphism of N. We shall construct a semigroup S, $S \supset T$, in which N is an ideal and every proper homomorphism of which also induces a proper homomorphism of N. This will prove the theorem.

Denote by the principal ideal generated by the element a in the semigroup T, through A. $A = a \cup Ta \cup aTaT$. From Lemma 1 it follows that $A(N)$ is an ideal in T; therefore $A \subseteq A(N)$.

We shall put the elements of the set A in a completely ordered form in an arbitrary way by means of the set $\mathfrak{A}$ of indices $0, 1, 2, \dots, \alpha, \alpha + 1, \dots$, putting only $a_0 = 0,\ a_1 = a$.

The semigroup T decomposes into the set-theoretic sum of pairwise disjoint subsets $K_{\alpha\beta}$, $\alpha, \beta \in \mathfrak{A}$, where $K_{\alpha\beta}$ is the set of all elements x of T satisfying the relations $xa = a_\alpha,\ ax = a_\beta$. It is obvious, in particular, that $N \subseteq K_{00}$. Some of these subsets may be empty (this may occur, for example, for those α and β which correspond to elements of TaT).

The sequel rests on the following lemmas, in which $\alpha, \beta, \gamma, \delta, \lambda, \mu$ are arbitrary indices from $\mathfrak{A}$ such that the corresponding complexes $K_{\alpha\beta}, K_{\gamma\delta}, K_{\lambda\beta}, K_{\mu\gamma}$ are nonempty.

Lemma 2. *The products $a_\alpha K_{\gamma\delta}$ and $K_{\alpha\beta} a_\delta$ are equal and are equal to an element $a_\tau \in A$, where τ is some function of α and δ, $\tau = \tau(\alpha, \delta)$, $\tau \in \mathfrak{A}$.*

Proof. The equality $a_\alpha K_{\gamma\delta} = K_{\alpha\beta} a K_{\gamma\delta} = K_{\alpha\beta} a_\delta$ is obvious. In view of the equalities $K_{\alpha\beta} a = K_{\alpha\beta'} a,\ aK_{\gamma\delta} = aK_{\gamma'\delta}$, where $\beta' \neq \beta,\ \gamma' \neq \gamma$, the independence of the product $K_{\alpha\beta} a K_{\gamma\delta}$ of β and γ is clear. It remains to show that the product $K_{\alpha\beta} a K_{\gamma\delta}$ is equal to one element of A. Indeed, let $k_{\alpha\beta}$ and $k'_{\alpha\beta}$ be arbitrary elements of $K_{\alpha\beta}$, and $k_{\gamma\delta}$ and $k'_{\gamma\delta}$ arbitrary elements of $K_{\gamma\delta}$. Then

$$k_{\alpha\beta} a k_{\gamma\delta} = k_{\alpha\beta} \cdot a k'_{\gamma\delta} = k_{\alpha\beta} a \cdot k'_{\gamma\delta} = k'_{\alpha\beta} a k'_{\gamma\delta}.$$

Thus, $K_{\alpha\beta} a K_{\gamma\delta} = a_{\tau(\alpha,\delta)}$.

Remark. Obviously,

$$\tau(\alpha, 0) = \tau(0, \beta) = 0, \qquad \tau(\alpha, 1) = \alpha, \qquad \tau(1, \beta) = \beta.$$

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[*] A more particular question (whether a nilpotent semigroup can be a densely embedded ideal) was posed to the author by L. M. Gluskin, to whom the author expresses his sincere gratitude. The main result of the present note gives a negative answer for a considerably broader class of semigroups.

Lemma 3. *Let $\tau(\alpha, \beta)$ be the function defined in Lemma 2. The inclusion*

$$K_{\alpha\beta}K_{\gamma\delta} \subseteq K_{\tau(\alpha,\gamma)\,\tau(\beta,\delta)}$$

holds.

Proof.

$$K_{\alpha\beta}K_{\gamma\delta}a = K_{\alpha\beta}a_\gamma = a_{\tau(\alpha,\gamma)}, \qquad aK_{\alpha\beta}K_{\gamma\delta} = a_\beta K_{\gamma\delta} = a_{\tau(\beta,\delta)}.$$

These two equalities prove the lemma.

Lemma 4. *Let $\tau(\alpha, \beta)$ be the function defined in Lemma 2. The equality*

$$\tau(\tau(\alpha, \beta), \gamma) = \tau(\alpha, \tau(\beta, \gamma)).$$

holds.

Proof.

$$a_{\tau(\tau(\alpha,\beta),\gamma)} = a_{\tau(\alpha,\beta)}K_{\mu\gamma} = a_\alpha K_{\alpha\beta}K_{\mu\gamma} = a_\alpha K_{\tau(\lambda,\mu)\,\tau(\beta,\gamma)} = a_{\tau(\alpha,\tau(\beta,\gamma))}.$$

This proves the lemma.

Remark. The function $\tau(\alpha, \beta)$ may, obviously, be regarded as a partially defined single-valued binary operation on the set $\mathfrak{A}$. Lemma 4 then asserts that this operation is associative.

We shall construct the semigroup S by adjoining to T the elements $b, h_1, h_2, \dots, h_\alpha, h_{\alpha+1}, \dots, g_1, g_2, \dots, g_\alpha, g_{\alpha+1}, \dots,$ specified by the following defining relations, reduced for clarity to a multiplication table (where we put $h_0 = g_0 = 0$):

	$K_{\gamma\delta}$	b	g_γ	h_γ
$K_{\alpha\beta}$	$K_{\tau(\alpha,\gamma)\,\tau(\beta,\delta)}$	h_α	0	$h_{\tau(\alpha,\gamma)}$
b	g_δ	a	a_γ	0
h_α	0	a_α	$a_{\tau(\alpha,\gamma)}$	0
g_α	$g_{\tau(\alpha,\delta)}$	0	0	0

Here the complexes $K_{\alpha\beta}, K_{\gamma\delta}$ are taken to be nonempty. The complex standing in the upper left corner is contained in the product of the corresponding complexes. In the remaining cases equalities hold. Let us note that we have deliberately interchanged the elements g_γ and h_γ in the row (unlike in the column), in order to emphasize a certain symmetry of the table, which we shall use later in the proof.

It is necessary to verify the correctness of the multiplication defined in this way, or, in other words, the validity of associativity. Since associativity holds in T, it is necessary to consider products in which at least one element belongs to

$S \setminus T$. Since all elements of some complex $K_{\alpha\beta}$, when multiplied on the right (on the left) by a certain element of $S \setminus T$, give one and the same element, this makes it possible to take the whole complex when considering products in which elements of T take part. Thus, we must show the equality of the two products $XY \cdot Z$ and $X \cdot YZ$, where at least one of the complexes X, Y, Z is an element of $S \setminus T$. In doing so, we shall constantly use Lemmas 2, 3, 4, without mentioning this in the appropriate places.

The following cases are possible:

A. Among the complexes being multiplied there are two from T.

1. $K_{\alpha\beta}K_{\gamma\delta} \cdot b = K_{\tau(\alpha,\gamma)\tau(\beta,\delta)}b = h_{\tau(\alpha,\gamma)} = K_{\alpha\beta}h_\gamma = K_{\alpha\beta} \cdot K_{\gamma\delta}b$.
2. $K_{\alpha\beta}K_{\gamma\delta} \cdot h_\varepsilon = K_{\tau(\alpha,\gamma)\tau(\beta,\delta)}h_\varepsilon = h_{\tau(\tau(\alpha,\gamma),\varepsilon)} = h_{\tau(\alpha,\tau(\gamma,\varepsilon))} = K_{\alpha\beta}h_{\tau(\gamma,\varepsilon)} = = K_{\alpha\beta} \cdot K_{\gamma\delta}h_\varepsilon$.
3. $K_{\alpha\beta}K_{\gamma\delta} \cdot g_\varepsilon = K_{\tau(\alpha,\gamma)\tau(\beta,\delta)}g_\varepsilon = 0 = K_{\alpha\beta} \cdot 0 = K_{\alpha\beta} \cdot K_{\gamma\delta}g_\varepsilon$.
4. $K_{\alpha\beta}b \cdot K_{\gamma\delta} = h_\alpha K_{\gamma\delta} = 0 = K_{\alpha\beta}\delta_\delta = K_{\alpha\beta} \cdot bK_{\gamma\delta}$.
5. $K_{\alpha\beta}h_\varepsilon \cdot K_{\gamma\delta} = h_{\tau(\alpha,\varepsilon)}K_{\gamma\delta} = 0 = K_{\alpha\beta} \cdot 0 = K_{\alpha\beta} \cdot h_\varepsilon K_{\gamma\delta}$.
6. $K_{\alpha\beta}g_\varepsilon \cdot K_{\gamma\delta} = 0 = K_{\alpha\beta} \cdot g_\varepsilon K_{\gamma\delta}$.

Thus, for all $x \in S \setminus T$, $K_{\alpha\beta}K_{\gamma\delta} \cdot x = K_{\alpha\beta} \cdot K_{\gamma\delta}x$ and $K_{\alpha\beta}x \cdot K_{\gamma\delta} = K_{\alpha\beta} \cdot xK_{\gamma\delta}$. From considerations of symmetry it is not difficult to conclude that, analogously to cases 1–3, for all $x \in S \setminus T$, $xK_{\alpha\beta} \cdot K_{\gamma\delta} = x \cdot K_{\alpha\beta}K_{\gamma\delta}$.

B. Among the complexes being multiplied there is one from T.

1. $K_{\alpha\beta}b \cdot b = h_\alpha b = a_\alpha = K_{\alpha\beta}a = K_{\alpha\beta} \cdot bb$.
2. $K_{\alpha\beta}h_\gamma \cdot b = h_{\tau(\alpha,\gamma)}b = a_{\tau(\alpha,\gamma)} = K_{\alpha\beta} \cdot a_\gamma = K_{\alpha\beta} \cdot h_\gamma b$.
3. $K_{\alpha\beta}g_\gamma \cdot x = 0 = K_{\alpha\beta} \cdot g_\gamma x$ for all $x \in S \setminus T$.
4. $K_{\alpha\beta}x \cdot h_\gamma = 0 = K_{\alpha\beta} \cdot xh_\gamma$ for all $x \in S \setminus T$.
5. $K_{\alpha\beta}b \cdot g_\gamma = h_\alpha g_\gamma = a_{\tau(\alpha,\gamma)} = K_{\alpha\beta} \cdot a_\gamma = K_{\alpha\beta} \cdot bg_\gamma$.
6. $K_{\alpha\beta}h_\gamma \cdot g_\delta = h_{\tau(\alpha,\gamma)}g_\delta = a_{\tau(\tau(\alpha,\gamma),\delta)} = a_{\tau(\alpha,\tau(\gamma,\delta))} = K_{\alpha\beta}a_{\tau(\gamma,\delta)} = K_{\alpha\beta} \cdot h_\gamma g_\delta$.

Thus, for arbitrary $x, y \in S \setminus T$, $K_{\alpha\beta}x \cdot y = K_{\alpha\beta} \cdot xy$. From considerations of symmetry it is not difficult to conclude that for arbitrary $x, y \in S \setminus T$, $xy \cdot K_{\alpha\beta} = x \cdot yK_{\alpha\beta}$. It is also not difficult to verify that for arbitrary $x, y \in S \setminus T$, $xK_{\alpha\beta} \cdot y = x \cdot K_{\alpha\beta}y = 0$.

C. All three complexes being multiplied are elements of $S \setminus T$.

Here, obviously, $xy \cdot z = x \cdot yz = 0$.

Taking into account the inclusion $N \subseteq K_{00}$ and using the remark to Lemma 2, it is easy to see that N will be an ideal in S.

We now show that every proper homomorphism φ of the semigroup S induces a proper homomorphism of N. Let $\varphi x = \varphi y;\ x \neq y;\ x, y \in S$. If $x, y \in T$, then φ induces a proper homomorphism of N by virtue of the condition. We shall therefore suppose that, for example, $x \notin T$. Consider the possible cases here. In doing so we use the following generally accepted notation: $A \to B$ means that from assertion A assertion B follows.

I. $x = b$.

1. $y \in T,\ y \in K_{\alpha\beta}$, a) $\alpha \neq 0 \to \varphi a_\alpha = \varphi ya = \varphi y \cdot \varphi a = \varphi b \cdot \varphi a = \varphi ba = \varphi 0$;

b) $\alpha = 0 \to \varphi a = \varphi b^2 = \varphi y \cdot \varphi b = \varphi yb = \varphi 0$, since $yb = 0$.

2. $y \in S \setminus T$. a) $y = h_\alpha$ (where $\alpha \neq 0$, since $y \notin T$) $\to \varphi a_\alpha = \varphi yb = = \varphi y \cdot \varphi b = \varphi y \cdot \varphi y = \varphi 0$;

b) $y = g_\alpha$ $(\alpha \neq 0) \to \varphi a_\alpha = \varphi by = \varphi b \cdot \varphi y = \varphi y^2 = \varphi 0$.

II. $x = g_\alpha$ $(\alpha \neq 0)$.

1. $y \in T,\ y \in K_{\gamma\delta}$.

a) $\delta \neq 0 \Rightarrow \varphi a_\delta = \varphi ay = \varphi a \cdot \varphi y = \varphi a \cdot \varphi g_\alpha = \varphi a g_\alpha = \varphi 0$;

b) $\delta = 0 \Rightarrow \varphi a_\alpha \cdot \varphi bx = \varphi b \cdot \varphi x = \varphi b \cdot \varphi y = \varphi by = \varphi 0$.

2. $y \in S \setminus T$.

a) $y = b$, the case was considered in I;

b) $y = h_\beta \Rightarrow \varphi a_\alpha = \varphi bx = \varphi b \cdot \varphi x = \varphi b \cdot \varphi y = \varphi by = \varphi 0$;

c) $y = g_\beta$ $(\beta \neq \alpha) \Rightarrow \varphi a_\alpha = \varphi bx = \varphi b \cdot \varphi x = \varphi by = \varphi a_\beta$.

III. $x = h_\alpha$ $(\alpha \neq 0)$. The case is analogous to the preceding one.

Thus, in all cases there are elements a_α and a_β, $a_\alpha \neq a_\beta$, for which $\varphi a_\alpha = \varphi a_\beta$, i.e. φ induces a proper homomorphism of the semigroup N, as was required to prove.

Let us note that it is impossible to establish, for the given class of semigroups, the nonfulfillment of the first of the requirements entering into the definition of a densely embedded ideal, as the following simple example shows. Let N be a zero semigroup, i.e. $N^2 = 0$, and let 0 be the zero of the semigroup N. Then N will be an ideal in the semigroup $T = N \cup e$, where e is an externally adjoined identity, and it is not difficult to verify that every proper homomorphism of T induces a proper homomorphism of N.

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Note: Figure translations are in progress. See original paper for figures.

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