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Abstract

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A CHARACTERISTIC DIFFERENCE SCHEME FOR A NONSTATIONARY KINETIC EQUATION

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The paper gives a method for constructing characteristic difference schemes for a nonstationary kinetic equation. The solution of the resulting system of equations is simplified by reduction to a difference scheme for the corresponding integral equation. The stability of the schemes obtained is determined only by the time step.

1. Neutron multiplication in the nonstationary regime of a reactor is characterized by the density of the number of neutrons $\psi(P, \mathbf{e}, v, t)$, where P is a point of the region G of three-dimensional Euclidean space R_3 (G is the region occupied by the reactor), $\mathbf{e}$ is the unit vector in the direction of the velocity, v is the neutron velocity, and t is time. The function ψ satisfies the linear integro-differential (kinetic) Boltzmann equation ([1–3]), which is a neutron balance equation. Numerous works are devoted to the approximate determination of ψ in stationary problems (a detailed bibliography is given in ([3,4])).

We shall consider a nonstationary problem in the one-group approximation. For the case of isotropic scattering, the equation for $\psi(P, \mathbf{e}, t)$ takes the form

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial \mathbf{e}} + \alpha\psi - \beta n = q(P, \mathbf{e}, t). \tag{1}$$

Here $v = 1$; $\partial\psi/\partial\mathbf{e}$ is the derivative in the direction $\mathbf{e}$; α is the absorption coefficient; β is the multiplication coefficient; $n(P, t) = \frac{1}{4\pi}\int \psi(P, \mathbf{e}, t)\, d\mathbf{e}$ is the mean density of the number of neutrons; $q(P, \mathbf{e}, t)$ is the density of neutron sources. Methods for calculating (1) are treated in ([5,6]). The solution is determined in a convex region G, bounded by a piecewise smooth surface S, under the initial and boundary conditions

$$\psi(P, \mathbf{e}, t)\big|_{t=t^0} = \psi_0(P, \mathbf{e}); \tag{2}$$

$$\psi(P, \mathbf{e}, t)\big|_{P\in S} = 0 \quad \text{for } (\mathbf{en}) < 0, \tag{3}$$

where $\mathbf{n}$ is the outward normal to S at the point P. Questions of existence and uniqueness of the solution of equation (1) are considered in ([2]).

2. We shall consider the solution of equation (1) in the six-dimensional phase space ([4]) $R_3 \times \Omega \times \Theta$, where R_3 is three-dimensional Euclidean space; Ω is the space formed by the unit vectors $\mathbf{e}$; Θ is a one-dimensional space $(t^0 \leq t < \infty)$.

An arbitrary point in phase space $R_3 \times \Omega \times \Theta$ will be denoted by $\widetilde{P} = (P, \mathbf{e}, t)$, $P \in R_3$, $\mathbf{e} \in \Omega$, $t \in \Theta$.

We shall consider the solution of equation (1) in the phase region $\widetilde{G} = G \times \Omega \times [t^0, T]$ of the phase space $R_3 \times \Omega \times \Theta$. In the phase space $R_3 \times \Omega$, we shall call the corresponding region $\overline{G} = G \times \Omega$, and the corresponding point $\overline{P} = (P, \mathbf{e})$.

The trajectories of neutrons between collisions coincide with the characteristics of the differential operator $L = \partial/\partial t + \partial/\partial \mathbf{e}$. Let two points in $\widetilde{G}$, $\widetilde{P}_2$ and $\widetilde{P}_1$, lie on one characteristic, i.e.

$$P_1 = P_2 - \mathbf{e}_2\tau, \qquad \tau = t^2 - t^1, \qquad \mathbf{e}_1 = \mathbf{e}_2. \tag{4}$$

Integrating (1) from $\widetilde{P}_1$ to $\widetilde{P}_2$ along the characteristic, we obtain

$$\mathcal{L}_{\widetilde{P}_2,\tau}\psi = Q_{\widetilde{P}_2,\tau}, \tag{5}$$

where

$$\mathcal{L}_{\widetilde{P}_2,\tau}\psi = \psi(\widetilde{P}_2) - \psi(\widetilde{P}_1)\exp\left(-\int_{\widetilde{P}_1}^{\widetilde{P}_2}\alpha\, d\xi\right) - \int_{\widetilde{P}_1}^{\widetilde{P}_2}\beta n(P')\exp\left(-\int_{P'}^{\widetilde{P}_2}\alpha\, d\xi\right) ds, \tag{6}$$

$$Q_{\widetilde{P}_2,\tau} = \int_{\widetilde{P}_1}^{\widetilde{P}_2} q(p')\exp\left(-\int_{P'}^{\widetilde{P}_2}\alpha\, d\xi\right) ds. \tag{7}$$

Integrating (5) with respect to $\mathbf{e}_2$ leads to the nonstationary Peierls equation ([1]).

3. On the time interval $[t^0, T]$ introduce the difference grid $H_t(t^0, t^1, \ldots, t^j, t^{j+1}, \ldots, t^N = T)$. In the domain $\overline{\widetilde{G}}$ define the grid $\widetilde{H}_\rho$ as the product of the grid H_ρ in the domain G and H_Ω in the space Ω. The grids H_ρ and H_Ω divide, respectively, G and Ω into cells. We include boundary surfaces among

the coordinate surfaces forming the grid H_ρ. The nonstandard form of boundary cells must be taken into account by special interpolation.

The general grid $\widetilde{H}_\rho$ in $\widetilde{G}$ is equal to $\overline{H}_\rho \times H_t$. Denote the nodes of the grid H_ρ by $P_1, P_2, \dots, P_i, \dots, P_I$, the nodes of the grid $\overline{H}_\rho$ by $\overline{P}_{ik}(P_i, \mathbf{e}_k)$, and those of the grid $\widetilde{H}_\rho$ by $\widetilde{P}^j_{ik}(P_i, \mathbf{e}_k, t^j)$. In a number of cases in Ω one uses a local coordinate system associated with the position of the point P. In this case the vector $\mathbf{e}_k$ depends on P_i, $\mathbf{e}_k = \mathbf{e}_k(P_i)$ (for example, in the case of spherical symmetry). For the grid $\overline{H}_\rho$ define the diameter ρ so that, as $\rho \to 0$, all dimensions of all cells tend to zero. To each point $\widetilde{P}^{j+1}_{ik}$ there corresponds a point $\widetilde{P}^j_{(ik)}$, determined according to (4). Obviously, the point $\widetilde{P}^j_{(ik)}$ is not (in general) a nodal point of the grid $\widetilde{H}_\rho$.

We shall call a function defined only at the nodes of $\widetilde{H}_\rho$ a grid function ψ^j_{ik}. Associate with it, by interpolation, the functions ψ_H and n_H:

$$\psi^j_H(\overline{P}) = \sum \zeta_{ik}(\overline{P})\psi^j_{ik}, \tag{8}$$

where $\overline{P}_{ik}$ are the vertices of the cell $\overline{G}_H$ in which $\overline{P}$ lies; $\zeta_{ik}(\overline{P})$ are nonnegative interpolation coefficients, chosen so that

$$\sum \zeta_{ik}(\overline{P}) = 1.$$

At the nodes of the grid H_ρ define n^j_H by integrating ψ^j_H:

$$n^j_H(P_i) = \frac{1}{4\pi}\int \psi_H(P_i, \mathbf{e})\, d\mathbf{e} = \sum \gamma_k \psi^j(\overline{P}_{ik}). \tag{9}$$

Extend n_H outside the nodes on G and on S by interpolation of the type (8).

To determine the function n_H on characteristics for $t^j \le t \le t^{j+1}$, define n_H at the points where the characteristic intersects the faces G_H, interpolating in t between n^j_H and n^{j+1}_H. On the segment of a characteristic lying in a cell G_H, n_H is determined by interpolation along the characteristic from the values at the ends of the segment.

The adopted interpolation process assigns to any discontinuous function $\psi(\widetilde{P})$ the function ψ_H for the chosen grid $\widetilde{H}_\rho$. In this case the grid function is $\psi^j_{ik} = \psi(\widetilde{P}^j_{ik})$. From the function ψ_H at the nodes of the grid $\widetilde{H}$ one can compute $\mathcal{L}_{\widetilde{P}^{j+1}_{ik}, \tau^j}\psi_H$ by formula (6). It is obvious that in this case $\mathcal{L}\psi_H$ is determined through ψ^j_{ik}, i.e., the operator $\mathcal{L}_{\widetilde{P}^{j+1}_{ik}, \tau^j}\psi_H$ is a difference operator.

The accuracy of the adopted interpolation on the chosen grid for the function ψ determines the accuracy of the local approximation of the difference operator

$$\mathcal{E}_{\psi}(\tau,\rho)=\sup\left|\frac{1}{\tau^{j}}\mathcal{L}_{\widetilde{P}_{ik}^{j+1},\tau^{j}}(\psi-\psi_{H})\right|. \tag{10}$$

To determine the approximate grid function and the corresponding function ψ_H, we require that equation (5) be satisfied for the function ψ_H at the nodal points of the grid.

To determine ψ_H^{j+1} and n_H^{j+1} at the nodal points, given ψ_H^j and n_H^j, we obtain the system of equations

$$\mathcal{L}_{\widetilde{P}_{ik}^{j+1},\tau^{j}}\psi_{H}=Q_{\widetilde{P}_{ik}^{j+1},\tau^{j}}, \tag{11}$$

$$n_{H}^{j+1}(P_{i})=\sum\nolimits^{\prime}\gamma_{k}\psi^{j+1}(\overline{P}_{ik}). \tag{12}$$

The resulting system of linear equations can be solved by the method of iterations; the number of unknowns is then equal to $\sim I\times K$. However, the resulting system of equations can be reduced to a system of linear equations only with respect to n_i^{j+1}, i.e., one obtains a difference scheme for the nonstationary Peierls integral equation. In this case the number of unknowns is $\sim I$.

Multiplying (11) by γ_k and summing, we obtain a system of linear equations for $n_H^{j+1}(P_i)$. If n_H on the characteristic is determined by interpolation between $n_H^{j+1}(P_i)$ and $n_H^j(P_i-e_k\tau^j)$, then the system of equations for determining n_H^{j+1} decouples, and each $n_H^{j+1}(P_i)$ is determined independently.

Convergence of the resulting difference scheme in the class of continuous solutions is ensured by the fulfillment of the maximum principle (stability) and by the local approximation (7). More precisely, the following estimates hold:

1) Maximum principle

$$\|\psi\|^{j+1}\leq\frac{\|\psi\|^{j}+\|Q\|^{j+1}\tau^{j}}{1-\omega\tau^{j}}\quad\text{for }\tau^{j}<\frac{1}{\omega},$$

where

$$\|\psi\|^{j}=\sup|\psi_{H}^{j}|,\qquad\omega=\sup\left(\frac{1}{\tau^{j}}\int_{\widetilde{P}_{(ik)}^{j}}^{\widetilde{P}_{ik}^{j+1}}\beta\exp\left(-\int_{P'}\alpha\,d\xi\right)\,ds\right),$$

$$\|Q\|^{j+1}=\sup\left|\frac{1}{\tau^{j}}Q_{\widetilde{P}_{ik}^{j+1},\tau^{j}}\right|.$$

2) The accuracy of the difference scheme is determined by the difference $v_H=\widetilde{\psi}_H-\psi_H$, where $\widetilde{\psi}_H$ is the interpolating function of the exact solution $\widetilde{\psi}$ of equation (5), while ψ_H is determined by the solution of the difference scheme. For the error v_H po-

we obtain the estimate $\sup |v_H|^j \leqslant \varepsilon^j$, where ε^j is defined by the recurrence formula

$$\varepsilon^j = \frac{\varepsilon^{j-1} + \varepsilon(\tau,\rho)\tau^j}{1-\omega\tau^j}.$$

For $\tau^j = \tau \to 0$, for a given t we obtain

$$\varepsilon(t) \sim \varepsilon(t^0)\exp\left[\omega(t-t^0)\left(1-\frac{\omega\tau}{2}+\frac{\omega^2\tau^2}{3}\right)\right]+$$

$$+\frac{\varepsilon(\tau,\rho)}{\omega}\left\{\exp\left[\omega(t-t^0)\left(1-\frac{\omega\tau}{2}+\frac{\omega^2\tau^2}{3}\right)\right]-1\right\}.$$

Here $\varepsilon(\tau,\rho) = \varepsilon_\psi(\tau,\rho)$ on the exact solution $\widetilde{\psi}$.

Proceeding from the definition of $\varepsilon(\tau,\rho)$, we obtain sufficient conditions for approximation and convergence

$$\text{a)} \quad \left|\frac{\widetilde{\psi}_H\left(\widetilde{P}_{ik}^{j+1}-e_k\tau\right)-\widetilde{\psi}\left(\widetilde{P}_{ik}^{j+1}-e_k\tau\right)}{\tau}\right| \leqslant \varepsilon_1(\tau,\rho), \quad \text{where } \varepsilon_1(\tau,\rho) \xrightarrow[\tau\to 0,\ \rho\to 0]{} 0,$$

$$\text{b)} \quad \left|\tilde{n}_H(\widetilde{P})-\tilde{n}(\widetilde{P})\right| \leqslant \varepsilon_2(\tau,\rho), \quad \text{where } \varepsilon_2(\tau,\rho) \xrightarrow[\tau\to 0,\ \rho\to 0]{} 0.$$

For the difference scheme obtained, as for the original equation, there is continuous dependence on the parameters α, β and on the sources q, by virtue of the fulfillment of the maximum principle. Therefore, in the difference scheme the exact values α, β, and q may be replaced by mean values over the mesh. A generalization of the method to the case of stationary discontinuities of the solution and weak anisotropy of scattering is possible.

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Note: Figure translations are in progress. See original paper for figures.

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