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Abstract

Full Text

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ON TABLES AND INTERPOLATION OF FUNCTIONS FROM A CERTAIN CLASS

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The class of functions $E_s^\alpha(C)$ consists of all functions $f(x_1, \dots, x_s)$,

$$f(x_1, \dots, x_s) = \sum_{m_1, \dots, m_s = -\infty}^{\infty} C(m_1, \dots, m_s) e^{2\pi i (m_1 x_1 + \dots + m_{sx} s)},$$

whose Fourier coefficients $C(m_1, \dots, m_s)$ satisfy the inequality

$$|C(m_1, \dots, m_s)| \le \frac{C}{(\bar{m}_1 \dots \bar{m}_s)^\alpha},$$

where $\bar{m}_\nu = \max(|m_\nu|, 1)$, $\alpha > 1$, and C does not depend on $m_1, \dots, m_s$.

Lemma 1. Let $f \in E_s^\alpha(C)$. For every prime $N > s$ one can specify integers $a_1, \dots, a_s$ such that

$$\left| \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_s)\, dx_1 \dots dx_s - \frac{1}{N} \sum_{k=1}^{N} f\left(\frac{ka_1}{N}, \dots, \frac{ka_s}{N}\right) \right| \le A_0 C N^{-\alpha} \ln^{\alpha s} N,$$

where $A_0 = A_0(\alpha, s)$*.

The **proof** is contained in the proof of Theorem 1 of the paper ([1]).

Lemma 2. If $f \in E_s^\alpha(C)$, $\bar{n}_1 \dots \bar{n}_s < N_1$, then

$$\varphi(x_1, \dots, x_s) \equiv f(x_1, \dots, x_s) e^{-2\pi i (n_1 x_1 + \dots + n_{sx} s)} \in E_s^\alpha(A_1 C N_1^\alpha).$$

Proof. We note that

$$\frac{\bar{m}}{\overline{(m-n)}\, \bar{n}} < A,$$

where A is a constant independent of m and n. Let $C(m_1, \dots, m_s)$ and $C'(m_1, \dots, m_s)$ be the Fourier coefficients of the functions f and φ, respectively. Then

$$|C'(m_1, \dots, m_s)| = |C(m_1 - n_1, \dots, m_s - n_s)| \le \frac{C}{(\overline{m_1 - n_1} \dots \overline{m_s - n_s})^\alpha} =$$

$$= \frac{C(\bar n_1 \dots \bar n_s)^\alpha}{(\bar m_1 \dots \bar m_s)^\alpha} \left[\frac{\bar m_1}{\overline{(m_1 - n_1)}\,\bar n_1} \cdots \frac{\bar m_s}{\overline{(m_s - n_s)}\,\bar n_s} \right]^\alpha \le \frac{CA^{\alpha s} N_1^\alpha}{(\bar m_1 \dots \bar m_s)^\alpha},$$

which proves the lemma.

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* Here and below $A_\nu = A_\nu(\alpha, s)$ denotes constants depending only on α and s.

Lemma 3. If $1 < N_1 < N,\ \alpha > 1$, then the estimates

$$\sum_{\bar m_1 \dots \bar m_s \ge N_1} (\bar m_1 \dots \bar m_s)^{-\alpha} \ll A_2 N_1^{-(\alpha-1)} \ln^{s-1} N,$$

$$\sum_{\bar m_1 \dots \bar m_s < N_1} 1 \ll A_3 N_1 \ln^{s-1} N$$

hold.

Proof. The lemma is proved elementarily by induction on s. Denote

$$N_1 = [\sqrt{N} \ln^{s/2} N],$$

$$\widetilde{C}(m_1, \dots, m_s) = \frac{1}{N} \sum_{k=1}^{N} f\left(\frac{ka_1}{N}, \dots, \frac{ka_s}{N}\right) e^{-2\pi i \, \frac{a_1 m_1 + \dots + a_s m s}{N} \, k} \qquad (1)$$

$$P(x_1, \dots, x_s) = \sum_{\bar m_1 \dots \bar m_s < N_1} \widetilde{C}(m_1, \dots, m_s) e^{2\pi i (m_1 x_1 + \dots + m_s x s)}.$$

The trigonometric polynomial (1) may be regarded as an interpolation polynomial for the function $f(x_1, \dots, x_s)$, constructed from the table of values of this function at the points

$$\left(\frac{ka_1}{N}, \dots, \frac{ka_s}{N}\right), \quad k = 1, 2, \dots, N.$$

The grid formed by these points in the space $x_1, \dots, x_s$ will be called parallelepipedal.

Theorem. If $f \in E_s^\alpha(C)$, then the estimate

$$\int_0^1 \cdots \int_0^1 |f(x_1, \ldots, x_s) - P(x_1, \ldots, x_s)|^2 \, dx_1 \cdots dx_s = O\Big(N^{-\alpha+1/2} \ln^{(\alpha+1/2)s-1} N\Big)$$

holds.

Proof. Applying abbreviated notation, we obtain

$$\int |f(\mathbf{x}) - P(\mathbf{x})|^2 \, d\mathbf{x} = \sum_{\bar{m}_2 \ldots \bar{m}_s < N_1} |C(\mathbf{m}) - \widetilde{C}(\mathbf{m})|^2 + \sum_{\bar{m}_1 \ldots \bar{m}_s \geq N_1} |C(\mathbf{m})|^2, \quad (2)$$

where

$$\widetilde{C}(\mathbf{m}) = \widetilde{C}(m_1, \ldots, m_s),$$

$$C(\mathbf{m}) = \int_0^1 \cdots \int_0^1 f(x_1, \ldots, x_s) e^{-2\pi i (x_1 m_1 + \cdots + x_{sm} s)} \, dx_1 \cdots dx_s.$$

By virtue of the definition of $\widetilde{C}(\mathbf{m})$, using Lemmas 1 and 2, we obtain

$$|C(\mathbf{m}) - \widetilde{C}(\mathbf{m})| \ll A_4 C N_1^{\alpha} N^{-\alpha} \ln^{\alpha s} N. \quad (3)$$

Observing that

$$|C(\mathbf{m})|^2 \ll C^2 (\bar{m}_1 \ldots \bar{m}_s)^{-2\alpha},$$

from (2) and (3), according to Lemma 3, we obtain

$$\int |f(\mathbf{x}) - P(\mathbf{x})|^2 \, d\mathbf{x} \ll A_5 C^2 \left(N_1^{2\alpha+1} N^{-2\alpha} \ln^{2\alpha s + s - 1} N + N_1^{-(2\alpha-1)} \ln^{s-1} N\right) =$$

$$= O\Big(N^{-(\alpha-1)/2} \ln^{(\alpha+1/2)s-1} N\Big),$$

where the constant in the O depends on α, s, and C.

Let us compare the tables of values of functions $f \in E_s^{\alpha}(C)$ at the points of a parallelepipedal grid with the tables of values of functions $f \in E_s^{\alpha}(C)$ at the nodes of a cubic grid whose sides are parallel to the coordinate axes and have length $h = 1/n$. The number of all grid points lying in the unit cube,

i.e., $N \cong n^s$. Any interpolation formula constructed from tables of values of functions $f \in E_s^{\alpha}(C)$ at the nodes of a cubic grid has a remainder term of order no greater than $O(N^{-2\alpha/s})$, which decreases with increasing s. To prove this assertion it suffices to note that the function $f_1(x_1, \dots, x_s) \equiv 0$ and the function

$$f_2(x_1, \dots, x_s) = 2C\frac{\sin n\pi x_1}{n^\alpha} \in E_s^\alpha(C)$$

have identical tables (identically zero), although

$$\int |f_1(\mathbf{x}) - f_2(\mathbf{x})|^2 \, d\mathbf{x} = \frac{2C^2}{n^{2\alpha}} = O\left(N^{-\frac{2\alpha}{s}}\right).$$

After the completion of this work it became known to me that, independently of me, S. A. Smolyak was engaged in closely related questions (see ([2])).

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CITED LITERATURE

[1] N. M. Korobov, DAN, **124**, No. 6, 1207 (1959).
[2] S. A. Smolyak, DAN, **131**, No. 5 (1960).

Note: Figure translations are in progress. See original paper for figures.