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## Abstract

## Full Text

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*MATHEMATICS*

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# ON THE CONSTRUCTION OF ENVELOPES OF HOLOMORPHY FOR DOMAINS OF A SPECIAL TYPE

(Presented by Academician N. N. Bogolyubov, 3 V 1960)

We shall denote by  $R^{n+1}$  the real space of dimension  $n + 1$ , and by  $x = (x_0, x_1, \dots, x_n) = (x_0, \mathbf{x})$ ,  $p, q, \dots$  its points; by  $C^{n+1} = R^{n+1} + iR^{n+1}$  the complex space consisting of points  $\zeta = (\zeta_0, \vec{\zeta}) = p + iq$ ; the scalar products  $\zeta x = \zeta_0 x_0 - \vec{\zeta} \mathbf{x}$ ,  $\vec{\zeta} \mathbf{x} = \sum_{1 \leq j \leq n} \zeta_j x_j$ ,  $x^2 = x_0^2 - \mathbf{x}^2$ ; the Euclidean distances  $|\mathbf{x}| = \sqrt{\mathbf{x}^2}$ ,  $|x| = \sqrt{x_0^2 + \mathbf{x}^2}$ . The future ( $x_0 > |\mathbf{x}|$ ) and past ( $x_0 < -|\mathbf{x}|$ ) light cones will be denoted respectively by  $\Gamma^+$  and  $\Gamma^-$ ,  $\Gamma = \Gamma^+ \cup \Gamma^-$ . The tube domains in  $C^{n+1}$  of the form  $R^{n+1} + i\Gamma^+$  and  $R^{n+1} + i\Gamma^-$  will be denoted respectively by  $T^+$  and  $T^-$ ,  $T = T^+ \cup T^-$ .

A smooth curve will be called **timelike** if its tangent vector belongs to  $\Gamma$ ; a smooth surface will be called **spacelike** if its normal belongs to  $\Gamma$ .

By a generalized function we mean any linear continuous functional on the Schwartz space  $S$  ((1), Ch. VII). The space of generalized functions (conjugate to  $S$ ) will be denoted by  $S^*$ . Generalized functions  $F^+(p)$  and  $F^-(p)$  will be called **retarded** and **advanced**, respectively, if their Fourier transforms

$$\tilde{F}^\pm(x) = \int F^\pm(p) e^{-ipx} dp, \quad px = p_0 x_0 - \mathbf{p} \mathbf{x}$$

vanish respectively outside  $\bar{\Gamma}^+$  and  $\bar{\Gamma}^-$ . A generalized function  $f(p)$  will be called a **commutator** if  $\tilde{f}(x) = 0$  for  $x^2 < 0$ .

We shall say that a function  $F(\zeta)$ , holomorphic in the domain  $T^+$  (respectively  $T^-$ ), belongs to the class  $N^+$  (respectively  $N^-$ ) if:

- 1) for any  $q \in \Gamma^+$  (respectively  $\Gamma^-$ ),  $\varepsilon > 0$ , and  $t \geq t_0 > 0$ , the estimate

$$|F(p + itq)| < c(q, \varepsilon, t_0) e^{\varepsilon t} (1 + |p|)^l$$

holds for some real  $l$ , depending only on  $F$ ;

- 2) in the space  $S^*$  there exists a boundary value  $F(p_0 + i0, \mathbf{p})$  (respectively  $F(p_0 - i0, \mathbf{p})$ ) of the function  $F(p + iq)$  as  $q \rightarrow 0$ ,  $q \in \Gamma^+$  (respectively  $\Gamma^-$ ).

We shall denote by  $N$  the intersection of the classes  $N^+$  and  $N^-$ . A function  $F(\zeta)$  of class  $N^+$  (respectively  $N^-$ ) is called an **analytic continuation** of the generalized function  $F(p_0 + i0, \mathbf{p})$  (respectively  $F(p_0 - i0, \mathbf{p})$ ).

Let  $G$  be an arbitrary open set in  $R^{n+1}$ , and let  $\tilde{G}$  be a complex neighborhood of  $G$  such that, if a ball of radius  $\eta$  belongs to  $G$ , then the corresponding complex ball of radius  $0.1\eta$  belongs to  $\tilde{G}$ .

**Theorem 1.** *In order that there exist a function  $F(\zeta)$  of class  $N$ , holomorphic in the domain  $T \cup \tilde{G}$ , it is necessary and sufficient that its boundary values  $F(p_0 + i0, \mathbf{p})$  and  $F(p_0 - i0, \mathbf{p})$ : 1) be respectively retarded and advanced functions, and 2) coincide in  $G$ .*

The necessity of condition 1) follows from the results of Schwarz<sup>(2)</sup>, Lions<sup>(3)</sup>, and N. N. Bogolyubov and O. S. Parasyuk<sup>(4)</sup> (for  $n = 0$ ). The necessity of condition 2) is obvious.

**Sufficiency.** From condition 1) it follows<sup>(2-4)</sup> that the functions  $F(p_0 \pm 0, \mathbf{p})$  admit analytic continuations  $F^\pm(\zeta)$  of class  $N^\pm$ , respectively, in the domains  $T^\pm$ . From condition 2), by virtue of the “edge of the wedge” theorem\*, it follows that the functions  $F^\pm(\zeta)$  are holomorphic in  $\tilde{G}$  and, consequently, determine a single function  $F(\zeta)$ , holomorphic in the domain  $T \cup \tilde{G}$ , of class  $N$ .

The function  $F(\zeta)$  is the simultaneous analytic continuation of the generalized functions  $F(p_0 \pm i0, \mathbf{p})$ .

The domain  $T \cup \tilde{G}$  in Theorem 1 is not a domain of holomorphy. Therefore the function  $F$  is holomorphic in a wider domain—in the envelope of holomorphy  $\mathcal{E}(T \cup \tilde{G})$ . Hence, in particular, it follows that the set on which the functions  $F(p_0 \pm i0, \mathbf{p})$  coincide cannot be arbitrary: it must be a real section of a certain domain of holomorphy. Theorems 2 and 3 are closely connected with this fact.

**Theorem 2.** *If the commutator  $f(p)$  vanishes in the domain  $G$ , then it also vanishes in the least convex envelope  $B_0(G)$  of the domain  $G$  with respect to timelike curves.*

**Proof.** Construct the advanced and retarded functions\*\*

$$F^+(p) = \int \theta(x_0) \tilde{f}(x) e^{ipx} dx, \quad F^-(p) = - \int \theta(-x_0) \tilde{f}(x) e^{ipx} dx,$$

where  $\theta(\xi) = \frac{1}{2}(1 + \text{sign } \xi)$ . Then

$$F^+(p) - F^-(p) = (2\pi)^{n+1} f(p).$$

By Theorem 1 there exists a holomorphic function  $F(\zeta)$  of class  $N$  in  $T \cup \widetilde{G}$  such that, in the space  $S^*$ , the limiting relations

$$F(p_0 + i\varepsilon, \mathbf{p}) \rightarrow F^+(p), \quad F(p_0 - i\varepsilon, \mathbf{p}) \rightarrow F^-(p) \quad \text{as } \varepsilon \rightarrow +0$$

hold.

It is therefore enough to prove that

$$B_0(G) \subset \text{Re } \mathcal{E}(T \cup \widetilde{G}).$$

Let  $p_1$  and  $p_2$  be arbitrary points of the domain  $G$  such that there exists a timelike curve connecting them.

First suppose that this curve lies entirely in  $G$ , and let us prove that all timelike curves connecting the points  $p_1$  and  $p_2$  lie entirely in  $\mathcal{E}(T \cup \widetilde{G})$ . It may happen that all timelike curves connecting  $p_1$  and  $p_2$  lie in  $G$ . Allowing for the other possibility, construct a family of timelike curves  $C_\alpha$ , depending continuously on the parameter  $\alpha$  ( $0 \leq \alpha \leq \alpha_0$ ) and connecting the points  $p_1$  and  $p_2$ , with  $C_\alpha$  ( $\alpha > 0$ ) lying entirely inside  $G$ , while  $C_0$  contains boundary points of the domain  $G$  forming a compact set  $K$ . The curves  $C_\alpha$  may be assumed analytic, given locally by the equations

$$p_j = p_{j,\alpha}(\xi), \quad j = 0, 1, \dots, n, \quad 0 \leq \alpha \leq \alpha_0, \quad 0 \leq \xi \leq 1.$$

Continue the functions  $p_{j,\alpha}(\xi)$  analytically to complex values  $\lambda = \xi + i\eta$ . By the Heine-Borel lemma, one may assume that the continued functions  $p_{j,\alpha}(\lambda)$  are holomorphic in the rectangle

$$\pi_\delta: \quad |\eta| \leq \delta, \quad 0 \leq \xi \leq 1$$

for sufficiently small  $\delta$ . Thus a family of two-dimensional analytic surfaces  $F_\alpha$  has been constructed,

$$\zeta_j = p_{j,\alpha}(\lambda), \quad j = 0, 1, \dots, n, \quad 0 \leq \alpha \leq \alpha_0, \quad \lambda \in \pi_\delta,$$

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\* This theorem was first proved by N. N. Bogolyubov (<sup>5</sup>), p. 152. For other proofs see (<sup>6-8</sup>).

\*\* The operation of multiplication by  $\theta(x_0)$  of a function of type  $\tilde{f}(x)$  can be defined up to a finite linear combination of  $\delta^{(k)}(x)$ . Therefore the functions  $F^\pm(p)$  are defined up to a polynomial.

passing through the curves  $C_\alpha$ . For sufficiently small  $\delta$ , all points  $\zeta$  on  $F_\alpha$  for which  $\lambda \in \pi_\delta$  and  $\eta \neq 0$  belong to  $T$ . Indeed, expanding  $p_{j,\alpha}(\lambda)$  in a Taylor series in a neighborhood of the point  $\lambda = \xi \in C_\alpha$ , we obtain

$$\zeta_j = p_{j,\alpha}(\xi) + p'_{j,\alpha}(\xi)(\lambda - \xi) + O[(\lambda - \xi)^2],$$

whence  $\text{Im } \zeta_j = \eta p'_{j,\alpha}(\xi) + O(\eta\delta)$ . But the curves  $C_\alpha$  are time-like. Therefore  $\text{Im } \zeta \in \Gamma$  if  $\eta \neq 0$ . If, however,  $\eta = 0$ , then, by construction,  $C_\alpha \subset G$  for  $\alpha > 0$  and  $C_0 \setminus K \subset G$ . By the continuity theorem (see <sup>(9)</sup>)  $K \subset \mathcal{E}(T \cup \tilde{G})$ , and, consequently, the curve  $C_0$  is entirely contained in  $\text{Re } \mathcal{E}(T \cup \tilde{G})$ .

Let us consider the other possibility, when there is no time-like curve connecting the points  $p_1$  and  $p_2$  and lying entirely in the domain  $G$ . Connect these points by some curve  $C$  lying entirely in  $G$ , and for all pairs of points on  $C$  that can be connected by a time-like curve lying entirely in  $G$ , carry out the extension described above. It is not hard to see that, in the end, the original points  $p_1$  and  $p_2$  will also be included among these pairs. The theorem is proved.

Thus it has been proved that  $B_0(G) \subset \text{Re } \mathcal{E}(T \cup \tilde{G})$ .

We give three examples of applications of Theorem 2.

- 1) If  $G$  is  $p^2 < -b^2$ , then  $B_0(G)$  is  $p^2 < 0$ ; this result was established by another method in <sup>(10)</sup>.
- 2) If  $G$  is  $p^2 > -b^2$ , then  $B_0(G) = R^{n+1}$  and the function  $F(\zeta)$  is a polynomial.
- 3) There are no commutators with compact support (except zero).

**Theorem 3.** *Every function  $F(\zeta)$  of class  $N$ , holomorphic in the domain  $T \cup \tilde{G}$ , where  $G$  is an arbitrary domain, is holomorphic in the domain  $H(T \cup \tilde{G})$ , consisting of those and only those points  $\zeta$  for which every complex hyperboloid  $(\zeta' - u)^2 = s$  ( $u, s \geq 0$  are real parameters), passing through  $\zeta$ , has with  $G$  at least one common interior point.*

For the proof we introduce the function (cf. <sup>(11, 12)</sup>)

$$u(p, k) = \frac{1}{(2\pi)^{n+1}} \int \tilde{f}(x) \cos(k\sqrt{x^2}) e^{ipx} dx.$$

Then  $u \in S^*$ ,

$$\left( \frac{\partial^2}{\partial p_0^2} - \frac{\partial^2}{\partial p_1^2} - \dots - \frac{\partial^2}{\partial p_n^2} - \frac{\partial^2}{\partial k^2} \right) u = 0, \quad u \Big|_{k=0} = f(p), \quad \frac{\partial u}{\partial k} \Big|_{k=0} = 0. \quad (1)$$

“Smoothing”  $u(p, k)$  with respect to  $p$ , we obtain an infinitely differentiable function satisfying the relations (1) with the “smoothed” function  $f(p)$ . Using Holmgren’s theorem and applying Theorem 2, we conclude that

$$u(p, k) = 0 \quad \text{for} \quad (p, k) \in B_0(G'),$$

where  $G'$  is some  $(n + 2)$ -dimensional neighborhood of the domain  $G$  lying in the plane  $k = 0$ . With the aid of (2) we determine the position of the support of the weight function in the integral representation of Jost–Lehmann–Dyson (see <sup>(8, 11, 12)</sup>) for the commutator  $F(p_0 + i0, p) - F(p_0 - i0, p)$ , and then use known techniques <sup>(8, 13)</sup>.

**Corollary.** If the commutator  $f(p)$  vanishes in the domain  $G$ , then it also vanishes in the domain  $B(G)$  obtained by bending  $G$  by the hyperboloids  $(p - u)^2 = s$ ,  $s \geq 0$ , i.e.

$$B(G) = \text{Re } H(T \cup \tilde{G}).$$

The domain  $H(T \cup \tilde{G})$  is pseudoconvex.

**Remark.** Theorem 3 is also valid for such open sets  $G$  in which all points of distinct connected components  $G_j$  are dis-

separated by space-like intervals, i.e.,  $(x_i - x_j)^2 < 0$ , if  $x_i \in G_i$ ,  $x_j \in G_j$ ,  $j \neq i$ .

Theorem 3 generalizes previously known results <sup>(8, 11–13)</sup> for domains  $G$  bounded by two space-like surfaces to the case of arbitrary domains. Dyson <sup>(11)</sup> stated a conjecture equivalent to the assertion that Theorem 3 is valid for any open set  $G$ .

The inclusion  $\mathcal{E}(T \cup \tilde{G}) \subset H(T \cup \tilde{G})$  is valid, since  $H(T \cup \tilde{G})$  is the envelope of holomorphy of  $T \cup \tilde{G}$  with respect to holomorphic functions of class  $N$ , whereas  $\mathcal{E}(T \cup \tilde{G})$  is the envelope of holomorphy of  $T \cup \tilde{G}$  with respect to all holomorphic functions. It is not clear whether  $\mathcal{E}(T \cup \tilde{G}) = H(T \cup \tilde{G})$ . If, for example,  $G$  is the strip  $|p_0| < a$ , then this is indeed so <sup>(5, 7)</sup>,

$$\mathcal{E}(T \cup \tilde{G}) = H(T \cup \tilde{G}) : |q| < \left| \text{Im} \sqrt{\xi_0^2 - a^2} \right|,$$

for in this case the domain  $T \cup \tilde{G}$  is semitubular <sup>(14)</sup>.

In conclusion I express my gratitude to N. N. Bogolyubov for his unfailing interest in the work and for discussion of the results.

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*Note: Figure translations are in progress. See original paper for figures.*

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