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Abstract

Full Text

MATHEMATICS

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PRIMARILY FACTORIZABLE GROUPS

(Presented by Academician A. I. Mal'cev on 23 IV 1960)

A subgroup \mathfrak{A} of a group \mathfrak{G} is called **complemented** in \mathfrak{G} if in \mathfrak{G} there exists a subgroup \mathfrak{B} such that $\mathfrak{A}\mathfrak{B} = \mathfrak{G}$ and $\mathfrak{A} \cap \mathfrak{B} = 1$. The condition that certain subgroups of a group be complemented sometimes determines its structure to a significant extent. For example, N. V. Baeva-Chernikova ^(1,2), studying arbitrary groups in which all subgroups are complemented (following her, we shall call such groups **fully factorizable**), showed that they are exhausted by certain periodic subgroups of complete direct products of finite groups whose orders are not divisible by squares of prime numbers, and even by complete direct products of specially chosen groups of this kind (called quite primitive by the author ⁽¹⁾). Fully factorizable groups have been studied completely in ⁽²⁾.

Groups with various kinds of systems of complemented subgroups under various additional conditions were considered by S. N. Chernikov in ⁽³⁾; in that work the general problem of studying groups with prescribed systems of complemented subgroups was first posed. In the present paper we solve the problem, proposed to the author by S. N. Chernikov, on the structure of groups with complemented p -subgroups for every prime p .

1. We shall call a group \mathfrak{G} **primarily factorizable** if all its p -subgroups are complemented in it for every prime number p .

Let \mathfrak{P} be a group of prime order p . A fully factorizable subgroup \mathfrak{A} of the holomorph of the group \mathfrak{P} will be called **fully p -primitive** if it contains the group \mathfrak{P} . If there is no need to indicate the order of the group \mathfrak{P} , then the group \mathfrak{A} will simply be called **fully primitive**.

Let

$$\widetilde{\mathfrak{G}} = \prod_{\alpha \in \mathfrak{M}} \mathfrak{G}_{\alpha}$$

be the complete direct product of certain fully primitive groups \mathfrak{G}_{α} , and

$$\mathfrak{G} = \prod_{\alpha \in \mathfrak{M}} \mathfrak{G}_{\alpha}$$

the direct product of the same groups. Denote by

$$\widetilde{\mathfrak{G}}_p = \prod_{\alpha \in \mathfrak{M}_p} \mathfrak{G}_{\alpha}$$

the complete direct product of all fully p -primitive (for fixed p) groups \mathfrak{G}_α , $\alpha \in \mathfrak{M}$.

A set \mathfrak{N}_p of components of the elements of a set \mathfrak{N} of elements of the group $\widetilde{\mathfrak{G}}$ (not necessarily distinct) in the group $\widetilde{\mathfrak{G}}_p$ will be called the **p -projection of the set \mathfrak{N}** .

Some set \mathfrak{A} of elements of the group $\widetilde{\mathfrak{G}}$ will be called a **commutative set** if all its elements commute with one another. If to each element $A \in \mathfrak{A}$ there is put in correspondence some element G of the commutant \mathfrak{G}' of the group \mathfrak{G} , then the set $\widetilde{\mathfrak{A}}$ of elements GA will be called an **almost commutative set of the group \mathfrak{G}** .

Let \mathfrak{N} be some set of elements of the group $\widetilde{\mathfrak{G}}$. If all its p -projections \mathfrak{N}_p (for all possible p) are almost commutative in the groups $\widetilde{\mathfrak{G}}_p$, then \mathfrak{N} will be called a **primary set of elements of the group \mathfrak{G}** .

Consider the set of functions $\varphi(n), \psi(n), \dots$ of a natural argument n , taking natural values.

Suppose that the equality $\varphi(n) = \psi(n)$ holds for only a finite number of values of the argument n . We shall call such functions *finitely linked*. A set \mathfrak{M} of pairwise finitely linked functions will be called *maximal* if it is not contained in another set of pairwise finitely linked functions.

2. Consider the complete direct product

$$\widetilde{\mathfrak{G}} = \prod_{(k,l)} \mathfrak{G}_{kl}$$

of nonabelian groups \mathfrak{G}_{kl} of order $2p_{kl}$, where p_{kl} is a prime number, $k, l = 1, 2, \dots$

In each of the groups \mathfrak{G}_{kl} choose two elements A_{kl} and B_{kl} of the second order. In the group $\widetilde{\mathfrak{G}}$ choose the elements A_n , $n = 1, 2, \dots$, by the following rule: the component of the element A_n in the group \mathfrak{G}_{ns} , $s = 1, 2, \dots$, is equal to A_{ns} , and all the remaining components of the element A_n are equal to the identity, i.e.

$$A = A_{n1}A_{n2} \dots A_{ns} \dots$$

To every function φ from some (fixed) maximal set \mathfrak{M} of pairwise finitely linked functions we put in correspondence an element B_φ of the group $\widetilde{\mathfrak{G}}$ according to the following rule: the component of the element B_φ of the group $\widetilde{\mathfrak{G}}$ in the group $\mathfrak{G}_{s,\varphi(s)}$, $s = 1, 2, \dots$, is equal to the element $B_{s,\varphi(s)}$, and all the remaining components of the element B_φ are equal to the identity, i.e.

$$B_\varphi = B_{1,\varphi(1)}B_{2,\varphi(2)} \dots B_{s,\varphi(s)} \dots$$

By the elements A_n , $n = 1, 2, \dots$, and B_φ , $\varphi \in \mathfrak{M}$, generate a subgroup \mathfrak{G} of the group $\widetilde{\mathfrak{G}}$. If $p_{kl} = p_{rs}$ if and only if $k = r$ and $l = s$, then the group \mathfrak{G} is a periodic primarily factorizable group, but is not completely factorizable, since the commutant \mathfrak{G}' of the group \mathfrak{G} is not complemented in \mathfrak{G} .

Consequently, the class of periodic primarily factorizable groups is broader than the class of completely factorizable groups.

3. The structure of periodic primarily factorizable groups is described by the following two propositions.

Theorem 1. *A periodic subgroup \mathfrak{A} of the complete direct product*

$$\widetilde{\mathfrak{G}} = \prod_{\alpha \in \mathfrak{M}} \mathfrak{G}_\alpha$$

of completely primitive groups \mathfrak{G}_α is primarily factorizable if and only if each of its p -projections \mathfrak{A}_p is completely factorizable.

Theorem 2. *A periodic primarily factorizable group \mathfrak{H} is isomorphic to some subgroup $\widetilde{\mathfrak{H}}$, generated by a primary set of elements of the complete direct product*

$$\widetilde{\mathfrak{G}} = \prod_{\alpha \in \mathfrak{M}} \mathfrak{G}_\alpha$$

of completely primitive groups \mathfrak{G}_α .

A periodic subgroup $\widetilde{\mathfrak{H}}$ of the group $\widetilde{\mathfrak{G}}$, generated by an arbitrary primary set of elements of the group $\widetilde{\mathfrak{G}}$, is primarily factorizable.

I express my gratitude to S. N. Chernikov for his guidance in carrying out the present work.

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CITED LITERATURE

¹ N. V. Baeva, DAN, 97, No. 5, 877 (1953). ² N. V. Chernikova, Matem. sborn., 39, 273 (1956). ³ S. N. Chernikov, Matem. sborn., 35, issue 1, 93 (1954).

Note: Figure translations are in progress. See original paper for figures.

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