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Abstract

Full Text

MATHEMATICS

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ON ESTIMATES OF GREEN TENSORS FOR SOME BOUNDARY-VALUE PROBLEMS

(Presented by Academician V. I. Smirnov on 19 X 1959)

1. Consider the first boundary-value problem for the stationary Navier–Stokes system in a bounded three-dimensional domain Ω with boundary S :

$$\Delta \mathbf{v} = \text{grad } p + \mathbf{f}, \quad \text{div } \mathbf{v} = 0, \quad \mathbf{v}|_S = 0, \quad (1)$$

Odqvist ⁽¹⁾ developed the theory of potentials and constructed the Green tensor for problem (1). The fundamental singular solution has the form

$$v_{ij}(x, y) = \frac{1}{8\pi} \left[\frac{\delta_{ij}}{r_{xy}} + \frac{(x_i - y_i)(x_j - y_j)}{r_{xy}^3} \right], \quad q_j(x, y) = \frac{1}{4\pi} \frac{x_j - y_j}{r_{xy}^3}.$$

The Green tensor constructed by Odqvist is expressed in terms of the fundamental singular solution in the following way:

$$G_{ij}(x, y) = v_{ij}(x, y) - \Gamma_{ij}^j(x, y), \quad g_j(x, y) = q_j(x, y) - \gamma_j(x, y),$$

where Γ_{ij} , γ_j satisfy the system

$$\Delta_x \Gamma_{ij}(x, y) = \frac{\partial \gamma_j(x, y)}{\partial x_i}, \quad \frac{\partial \Gamma_{ij}(x, y)}{\partial x_i} = 0,$$

and the boundary condition

$$\Gamma_{ij}(\xi, y)|_{\xi \in S} = v_{ij}(\xi, y)|_{\xi \in S}.$$

For G_{ij} , $\partial G_{ij}/\partial x_m$, g_j , Odqvist obtained a number of estimates. A generalization of his results is the following theorem:

Theorem 1. *If $S \in \text{Lip}(1, h)$, $0 < h \leq 1$, then*

$$|G_{ij}(x, y)| \leq \frac{C}{r_{xy}}, \quad \left| \frac{\partial G_{ij}(x, y)}{\partial x_m} \right|, \quad |g_j(x, y)| \leq \frac{C}{r_{xy}^2},$$

$$\left. \begin{array}{l} \left| \frac{\partial G_{ij}(x, y)}{\partial x_m} - \frac{\partial G_{ij}(x', y)}{\partial x'_m} \right| \\ |g_j(x, y) - g_j(x', y)| \end{array} \right\} \leq \begin{cases} C \left(\frac{r_{xx'} |\ln r_{xx'}|}{R^3} + \frac{r_{xx'}^h}{R^2} \right), & h < 1, \\ C \left(\frac{r_{xx'} |\ln r_{xx'}|}{R^3} + \frac{r_{xx'} |\ln r_{xx'}|^2}{R^2} \right), & h = 1, \end{cases}$$

where C are constants depending on S ; $R = \min(r_{xy}, r_{x'y})$.

If, however, $S \in \text{Lip}(2, \lambda)$, $0 < \lambda \leq 1$, then

$$\left| \frac{\partial^2 G_{ij}(x, y)}{\partial x_l \partial x_m} \right|, \quad \left| \frac{\partial g_j(x, y)}{\partial x_l} \right| \leq \frac{C}{r_{xy}^3},$$

$$\left| \frac{\partial^2 G_{ij}(x, y)}{\partial x_l \partial x_m} - \frac{\partial^2 G_{ij}(x', y)}{\partial x'_l \partial x'_m} \right| \ll \begin{cases} C \left(\frac{r_{xx'} |\ln r_{xx'}|}{R^4} + \frac{r_{xx'} |\ln r_{xx'}|^2}{R^3} + \frac{r_{xx'}^\lambda}{R^2} \right), & \lambda < 1, \\ C \left(\frac{r_{xx'} |\ln r_{xx'}|}{R^4} + \frac{r_{xx'} |\ln r_{xx'}|^2}{R^3} + \frac{r_{xx'} |\ln r_{xx'}|^3}{R^2} \right), & \lambda = 1. \end{cases}$$

We note that exactly the same estimates are valid for the Green's function for the Dirichlet problem and for the Neumann function.

Since the solution of problem (1) is expressed by means of the Green tensor according to the formulas

$$v_i(x) = - \int_{\Omega} G_{ij}(x, y) f_j(y) dy, \quad p(x) = - \int_{\Omega} g_j(x, y) f_j(y) dy,$$

then, with the aid of Theorem 1, the following can be proved.

Theorem 2. If $S \in \text{Lip}(1, h)$ and $|\mathbf{f}| \ll C$, then

$$\left| \frac{\partial v}{\partial x_i} \right| \ll C, \quad \left. \begin{array}{l} \left| \frac{\partial v(x)}{\partial x_i} - \frac{\partial v(x')}{\partial x'_i} \right| \\ |p(x) - p(x')| \end{array} \right\} \ll \begin{cases} Cr_{xx'}^h, & h < 1, \\ Cr_{xx'} |\ln r_{xx'}|^2, & h = 1. \end{cases}$$

If, however, $S \in \text{Lip}(2, \lambda)$, $\mathbf{f} \in \text{Lip}(0, \alpha)$, then

$$\left| \frac{\partial^2 v}{\partial x_i \partial x_j} \right|, \quad \left| \frac{\partial p}{\partial x_i} \right| \ll C \|\mathbf{f}\|_{\text{Lip}(0, \alpha)},$$

$$\left. \begin{aligned} & \left| \frac{\partial^2 v(x)}{\partial x_i \partial x_j} - \frac{\partial^2 v(x')}{\partial x'_i \partial x'_j} \right| \\ & \left| \frac{\partial p(x)}{\partial x_i} - \frac{\partial p(x')}{\partial x'_i} \right| \end{aligned} \right\} \ll \begin{cases} C \|\mathbf{f}\|_{\text{Lip}(0,\alpha)} r_{xx'}^\lambda, & \lambda < \alpha, \\ C \|\mathbf{f}\|_{\text{Lip}(0,\alpha)} r_{xx'}^\alpha |\ln r_{xx'}|, & \alpha \leq \lambda < 1, \quad \alpha < \lambda \ll 1, \\ C \|\mathbf{f}\|_{\text{Lip}(0,1)} r_{xx'} |\ln r_{xx'}|^3, & \alpha = \lambda = 1. \end{cases}$$

A more general result is given in the note (2).

In addition, the following theorem is valid.

Theorem 3. If $S \in C^2$, then the inequality

$$\|\mathbf{v}\|_{W_p^2(\Omega)} \ll C \|\mathbf{f}\|_{L_p(\Omega)}$$

holds.

We note that an analogous estimate for $\|\mathbf{v}\|_{W_p^2(\Omega')}$, where Ω' is any strictly interior subdomain of the domain Ω , was obtained by O. A. Ladyzhenskaya.

2. In the study of the differential properties of the solution of stationary problems of magnetohydrodynamics, it becomes necessary to construct the Green tensor for the following problem

$$\text{rot } \mathbf{H} = \mathbf{j}, \quad \text{div } \mathbf{H} = 0, \quad H_n|_S = 0, \quad (2)$$

where \mathbf{j} is a prescribed solenoidal vector, $H_n|_S = H_i(\xi)n_i(\xi)|_{\xi \in S}$, and Ω is a simply connected domain.

We shall also consider the adjoint problem

$$\text{rot } \mathbf{E} = \mathbf{a}, \quad \text{div } \mathbf{E} = 0, \quad \mathbf{E}_\tau|_S = 0, \quad (3)$$

where \mathbf{a} is a solenoidal vector satisfying the boundary condition $a_n|_S = 0$, and $\mathbf{E}_\tau = \mathbf{E} - \mathbf{n}E_n$.

The Green tensors for problems (2), (3), $V_{ik}(x, y)$ and $U_{ik}(x, y)$, are solutions of the following problems:

$$\text{rot}_x V_k(x, y) = \text{grad}_x \frac{\partial N(x, y)}{\partial y_k} + e_k \delta(x - y), \quad \text{div}_x V_k(x, y) = 0,$$

$$V_{k\tau}(\xi, y)|_{\xi \in S} = 0, \quad V_k = (V_{1k}, V_{2k}, V_{3k}),$$

$$\operatorname{rot}_x U_k(x, y) = \operatorname{grad}_x \frac{\partial G(x, y)}{\partial y_k} + e_k \delta(x - y),$$

$$\operatorname{div}_x U_k(x, y) = 0, \quad U_{kn}(\xi, y)|_{\xi \in S} = 0, \quad U_k = (U_{1k}, U_{2k}, U_{3k}),$$

where $N(x, y)$ is the Neumann function, and $G(x, y)$ is the Green function for the Dirichlet problem.

It is easy to verify that

$$V_k(x, y) = \frac{1}{4\pi} \operatorname{rot}_x \frac{e_k}{r_{xy}} + V'_k(x, y) + \operatorname{grad}_x s_k(x, y),$$

where

$$V'_k(x, y) = \frac{1}{4\pi} \operatorname{rot}_x \int_S \frac{\partial N(\xi, y)}{\partial y_k} \frac{n(\xi)}{r_{\xi x}} dS_\xi$$

and $s_k(x, y)$ is the solution of the Dirichlet problem:

$$\Delta_x s_k(x, y) = 0, \quad s_k(\xi, y)|_{\xi \in S} = b_k(\xi, y)|_{\xi \in S},$$

where

$$b_k(\xi, y) = - \int_{M_0}^{M(\xi)} \left(\frac{1}{4\pi} \operatorname{rot}_\eta \frac{e_k}{r_{\eta y}} + V'_k(\eta, y), \overline{de_\eta} \right);$$

the integration is carried out along a contour lying on S . The expression for $b_k(\xi, y)$ is meaningful, since

$$\oint_{(l)} \left(\frac{1}{4\pi} \operatorname{rot}_\eta \frac{e_k}{r_{\eta y}} + V'_k(\eta, y), \overline{dl_\eta} \right) = 0.$$

Furthermore,

$$U_k(x, y) = \frac{1}{4\pi} \operatorname{rot}_x \frac{e_k}{r_{xy}} + U'_k(x, y) + \operatorname{grad}_x t_k(x, y),$$

where

$$U'_k(x, y) = \operatorname{rot}_x \int_S \frac{\partial \rho(\xi, y)}{\partial y_k} \frac{n(\xi)}{r_{\xi x}} dS_\xi,$$

$\rho(\xi, y)$ is the density of the double-layer potential by means of which the regular part of the function $G(x, y)$ is expressed, and $t_k(x, y)$ is the solution of the Neumann problem

$$\Delta t_k(x, y) = 0,$$

$$\left. \frac{\partial t_k(\xi, y)}{\partial n_\xi} \right|_{\xi \in S} = -\frac{1}{4\pi} \operatorname{rot}_n \frac{e_k}{r_{\xi y}} + U'_{kn}(\xi, y) \Big|_{\xi \in S}.$$

Between V_{ik} and U_{ik} there exists the relation

$$V_{ik}(x, y) = U_{ki}(y, x).$$

The solutions of problems (2), (3) are expressed by means of V_{ik} and U_{ik} in the following way:

$$H_k(x) = \int_{\Omega} j_i(y) V_{ik}(y, x) dy = \int_{\Omega} U_{ki}(x, y) j_i(y) dy,$$

$$E_k(x) = \int_{\Omega} a_i(y) U_{ik}(y, x) dy = \int_{\Omega} V_{ki}(x, y) a_i(y) dy.$$

Theorem 4. If $S \in \operatorname{Lip}(1, h)$, then

$$|U_{ik}(x, y)|, |V_{ik}(x, y)| \leq \frac{C}{r_{xy}^2},$$

$$\left. \begin{array}{l} |U_{ik}(x, y) - U_{ik}(x', y)| \\ |V_{ik}(x, y) - V_{ik}(x', y)| \end{array} \right\} \leq \begin{cases} C \left(\frac{r_{xx'} |\ln r_{xx'}|}{R^3} + \frac{r_{xx'}^h}{R^2} \right), & h < 1, \\ C \left(\frac{r_{xx'} |\ln r_{xx'}|}{R^3} + \frac{r_{xx'} |\ln r_{xx'}|^2}{R^2} \right), & h = 1, \end{cases}$$

where $R = \min(r_{xy}, r_{x'y})$.

If, moreover, $S \in \operatorname{Lip}(2, \lambda)$, then

$$\left| \frac{\partial U_{ik}(x, y)}{\partial x_l} \right|, \left| \frac{\partial V_{ik}}{\partial x_l} \right| \leq \frac{C}{r_{xy}^3},$$

$$\left. \begin{aligned} \left| \frac{\partial U_{ik}(x, y)}{\partial x_l} - \frac{\partial U_{ik}(x', y)}{\partial x'_l} \right| \\ \left| \frac{\partial V_{ik}(x, y)}{\partial x_l} - \frac{\partial V_{ik}(x', y)}{\partial x'_l} \right| \end{aligned} \right\} \leq \begin{cases} C \left(\frac{r_{xx'} |\ln r_{xx'}|}{R^4} + \frac{r_{xx'} |\ln r_{xx'}|^2}{R^3} + \frac{r_{xx'}^\lambda}{R^2} \right), & \lambda < 1, \\ C \left(\frac{r_{xx'} |\ln r_{xx'}|}{R^4} + \frac{r_{xx'} |\ln r_{xx'}|^2}{R^3} + \frac{r_{xx'} |\ln r_{xx'}|^3}{R^2} \right), & \lambda = 1. \end{cases}$$

With the aid of these estimates the following can be proved.

Theorem 5. If $S \in \text{Lip}(1, h)$ and $|\mathbf{j}| \leq C$, then the solution of problem (1) satisfies the conditions:

$$|\mathbf{H}| \leq C, \quad |\mathbf{H}(x) - \mathbf{H}(x')| \leq \begin{cases} Cr_{xx'}^h, & h < 1, \\ Cr_{xx'} |\ln r_{xx'}|^2, & h = 1. \end{cases}$$

If, moreover, $S \in \text{Lip}(2, \lambda)$ and $\mathbf{j} \in \text{Lip}(0, \alpha)$, then

$$\left| \frac{\partial \mathbf{H}}{\partial x_l} \right| \leq C \|\mathbf{j}\|_{\text{Lip}(0, \alpha)},$$

$$\left| \frac{\partial \mathbf{H}(x)}{\partial x_l} - \frac{\partial \mathbf{H}(x')}{\partial x'_l} \right| \leq \begin{cases} C \|\mathbf{j}\|_{\text{Lip}(0, \alpha)} r_{xx'}^\lambda, & \lambda < \alpha, \\ C \|\mathbf{j}\|_{\text{Lip}(0, \alpha)} r_{xx'}^\alpha |\ln r_{xx'}|, & \alpha \leq \lambda < 1, \alpha < \lambda \leq 1, \\ C \|\mathbf{j}\|_{\text{Lip}(0, \alpha)} r_{xx'} |\ln r_{xx'}|^3, & \alpha = \lambda = 1. \end{cases}$$

An exactly analogous theorem is valid for problem 2.

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Note: Figure translations are in progress. See original paper for figures.

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