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1960

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Abstract

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ON THE INTERPOLATION OF STATIONARY PROCESSES WITH DISCRETE TIME

(Presented by Academician A. N. Kolmogorov, 25 IX 1959)

Let $x(t)$ be a stationary, in the broad sense, random process (the parameter t assumes integer values), whose realization is known at all times t, except for a finite number $\{t_1, t_2, \dots, t_s\} = T$; the note discusses the problem of linear interpolation of the quantities $x(t)$, $t \in T$, by means of the known part of the realization of the process $x(t)$, $t \notin T$.

As usual, we identify all random variables that differ from one another with probability equal to zero; then the totality of all random variables h with $M|h|^2 < \infty$ forms a Hilbert space with scalar product $(h_1, h_2) = Mh_1\overline{h_2}$. The quantity of the best linear interpolation of the unknown value $x(t)$, $t \in T$, is the projection $\hat{x}_T(t)$ of the quantity $x(t)$ onto the linear closure $\hat{H}(T)$ of the quantities $x(s)$, $s \notin T$, and the least error $\sigma_T(t)$ is equal to the "length" $\|h_T(t)\|$ of the perpendicular $h_T(t)$, dropped from $x(t)$ to $\hat{H}(T)$.

We shall say that the process $x(t)$ is **linearly interpolable** if $\sigma_T(t) \equiv 0$ for all T, t. As was shown in ([2,3]), for the interpolability of a process $x(t)$ with spectral density $f(\lambda)$, it is necessary and sufficient that

$$\int_{-\pi}^{\pi} \frac{|P(e^{i\lambda})|^2}{f(\lambda)}\, d\lambda = \infty \tag{1}$$

for any trigonometric polynomial $P(e^{i\lambda})$.

If condition (1) is not satisfied, then, as is not difficult to understand, there exists a **minimal** polynomial $P_0(e^{i\lambda})$, having the form

$$P_0(e^{i\lambda}) = \prod_{k=1}^{n} (e^{i\lambda} - e^{i\alpha_k})^{m_k}, \qquad \alpha_k \in [-\pi, \pi], \tag{2}$$

such that

$$\int_{-\pi}^{\pi} \frac{|P_0(e^{i\lambda})|^2}{f(\lambda)}\, d\lambda < \infty \tag{3}$$

and every polynomial $P(e^{i\lambda})$ for which

$$\int_{-\pi}^{\pi} \frac{|P(e^{i\lambda})|^2}{f(\lambda)}\, d\lambda < \infty$$

is divisible by $P_0(e^{i\lambda})$. We shall call the numbers α_k the zeros of the spectral density $f(\lambda)$ (in the case of a smooth spectral density $f(\lambda)$, the numbers α_k are its genuine zeros of multiplicity $2m_k$).

Denote by $H(T)$ the linear closure of the quantities $x(t)$, $t \in T$; set

$$\Delta(T) = H(T) \ominus \hat{H}(T). \tag{4}$$

As was shown in ([3]), the space $\Delta(T)$ is isometric to the space $\mathfrak{B}(T)$ of trigonometric polynomials $b(\lambda) = \sum_{t\in T} b_t e^{i\lambda t}$ for which

$$\int_{-\pi}^{\pi} \frac{|b(\lambda)|^2}{f(\lambda)}\, d\lambda < \infty,$$

with scalar product

$$(b_1, b_2) = \int_{-\pi}^{\pi} b_1(\lambda)\overline{b_2(\lambda)} \frac{d\lambda}{f(\lambda)}.$$

For simplicity, let the set T be an "interval" $\{t_0, t_0 + 1, \dots, t_0 + s - 1\}$ of length s.

From what was set forth above it follows:

Theorem 1. *All unknown values $x(t)$, $t \in T$, are interpolated without error ($|\Delta(T)| = 0$) if and only if the length s of the "interval" T does not exceed the number $m = \sum_{k=1}^{n} m_k$ of zeros of the spectral density $f(\lambda)$.*

In the case $s > m$, the dimension of the "error space" $\Delta(T)$ is equal to $s - m$.

Next, let

$$x(t) = \int_{-\pi}^{\pi} e^{i\lambda t} \Phi(d\lambda) \tag{5}$$

be the spectral representation of the process $x(t)$. The best interpolation quantity $\hat{x}_T(t)$ can be represented in the form

$$\hat{x}_T(t) = \int_{-\pi}^{\pi} \hat{\varphi}_T(\lambda, t) \Phi(d\lambda), \tag{6}$$

where the function $\hat{\varphi}_T(\lambda, t)$ is such that

$$\int_{-\varphi}^{\pi} |\hat{\varphi}_T(\lambda, t)|^2 f(\lambda)\, d\lambda < \infty, \tag{7}$$

and, moreover, $\hat{\varphi}_T(\lambda, t)$ can be approximated arbitrarily accurately in mean square with weight $f(\lambda)$ by trigonometric polynomials $\sum_{t_k \notin T} c_k e^{i\lambda t_k}$ ([4]); finding $\hat{x}_T(t)$ essentially amounts to finding the function $\hat{\varphi}_T(\lambda, t)$.

Consider the function

$$A_T(\lambda, t) = e^{i\lambda t} - \hat{\varphi}_T(\lambda, t). \tag{8}$$

Theorem 2. *The function $A_T(\lambda, t)$ has the form*

$$A_T(\lambda, t) = \frac{P_0(e^{i\lambda})}{f(\lambda)} \sum_{k=t_0-m}^{t_0-s+1} c_k e^{i\lambda k}, \tag{9}$$

where s is the length of the "interval" T; m is the degree of the minimal polynomial $P_0(e^{i\lambda})$; the coefficients c_k can be found from the following system of linear equations:

$$\sum_{k=t_0-m}^{t_0-s+1} A_{lk} c_k = B_l, \qquad t_0 - m \le l \le t_0 - s + 1, \tag{10}$$

$$A_{lk} = \int_{-\pi}^{\pi} e^{i\lambda(k-l)} \frac{|P_0(e^{i\lambda})|^2}{f(\lambda)}\, d\lambda,$$

$$B_l = \int_{-\pi}^{\pi} e^{i\lambda(t-l)} \overline{P}_0(e^{i\lambda})\, d\lambda.$$

For example, in the case when only one value $x(t)$ is unknown,

$$A_T(\lambda, t) = 2\pi \frac{e^{i\lambda t}}{f(\lambda)} \left(\int_{-\pi}^{\pi} \frac{d\mu}{f(\mu)} \right)^{-1} \tag{11}$$

and the error of the best interpolation is

$$\sigma_T(t) = 2\pi \left(\int_{-\pi}^{\pi} \frac{d\mu}{f(\mu)} \right)^{-1/2}. \tag{12}$$

Interpolability of the stationary process $x(t)$ means that

$$\hat{S} = \bigcap_T \hat{H}(T) = H, \tag{13}$$

where H is the linear closure of all quantities $x(t)$, $-\infty < t < \infty$.

Let us note that if the spectral function $F(\lambda)$ of the process $x(t)$ is not absolutely continuous, $F(\lambda) = \int_{-\pi}^{\lambda} f(\lambda) + \Sigma(\lambda)$, then the totality $\hat{S} = \bigcap_T \hat{H}(T)$ contains quantities s of the form

$$s = \int_{\Delta_0} s(\lambda)\Phi(d\lambda), \tag{14}$$

where Δ_0 is a set of Lebesgue measure zero on which the measure $d\Sigma(\lambda)$ is concentrated, and $\int_{\Delta_0} |s(\lambda)|^2 \, d\Sigma(\lambda) < \infty$ (see, for example, ([4])).

Theorem 3. *If the stationary process $x(t)$ is not interpolable, then the totality $\hat{S}$ consists of quantities of the form (14). In particular, if the spectral function $F(\lambda)$ of the process $x(t)$ is absolutely continuous, then $\hat{S} = 0$.*

Proof. Without loss of generality, one may assume that the spectral function $F(\lambda)$ of the process $x(t)$ is absolutely continuous (see, for example, ([4])). The space H in this case is isometric to the space $\mathfrak{A}$ of functions $a(\lambda)$,

$$\int_{-\pi}^{\pi} |a(\lambda)|^2 f(\lambda) \, d\lambda < \infty.$$

The quantities $\delta \in \Delta(T)$ will correspond to functions $\delta(\lambda) = b(\lambda)/f(\lambda)$, where

$$b(\lambda) = P_0(e^{i\lambda}) \sum_k e^{i\lambda k} \in \mathfrak{B}(T).$$

For any function $a(\lambda) \in \mathfrak{A}$ we have

$$\int_{-\pi}^{\pi} |a(\lambda) - \delta(\lambda)|^2 f(\lambda) \, d\lambda = \int_{-\pi}^{\pi} |a(\lambda)f(\lambda) - b(\lambda)|^2 \frac{d\lambda}{f(\lambda)} =$$

$$= \int_{-\pi}^{\pi} \left| \frac{a(\lambda)f(\lambda)}{P_0(e^{i\lambda})} - \sum_k c_k e^{i\lambda k} \right|^2 \frac{|P_0(e^{i\lambda})|^2}{f(\lambda)} \, d\lambda. \tag{15}$$

Since the function $g(\lambda) = |P_0(e^{i\lambda})|^2 f(\lambda)$ is integrable, and the function $\psi(\lambda) = a(\lambda)f(\lambda)/P_0(e^{i\lambda})$ is square-integrable with weight $g(\lambda)$, $\psi(\lambda)$ can be approximated arbitrarily accurately in the mean-square sense with weight $g(\lambda)$ by trigonometric polynomials $\sum_k c_k e^{i\lambda k}$. Therefore it follows from relation (15) that

$$\inf_{\delta} \int_{-\pi}^{\pi} |a(\lambda) - \delta(\lambda)|^2 f(\lambda)\, d\lambda = 0, \tag{16}$$

where the infimum is taken over all $\delta(\lambda)$ corresponding to quantities $\delta \in \Delta(T)$ for all possible T. Equality (16) means that $H = \bigcup_T \Delta(T)$, or $\bigcap_T \hat{H}(T) = 0$, as was required to prove.

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Received
1 VIII 1959

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