Reports of the Academy of Sciences of the USSR

1960

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR

1960. Vol. 131, No. 5

MATHEMATICS

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ON THE APPLICATION OF THE METHOD OF VARIATION OF A PARAMETER TO THE COMPUTATION OF EIGENVALUES AND EIGENVECTORS OF MATRICES

(Presented by Academician N. N. Bogolyubov on 16 XI 1959)

1°. Let there be given a square matrix $A(\lambda) = \|a_{kj}(\lambda)\|$ $(k, j = 1, 2, \dots, n)$ of order n, whose elements are functions of the parameter λ, taking prescribed values on some finite interval $\lambda_0 \le \lambda \le \lambda^*$.

By the eigenvalues of the matrix $A(\lambda)$ we shall, as usual, mean the solutions $p_i = p_i(\lambda)$ $(i = 1, 2, \dots, n)$ of its characteristic equation

$$\mathrm{Det}\, \|A(\lambda) - pE\| = 0.$$

To each eigenvalue $p_i(\lambda)$ we assign a nonzero eigenvector $X_i = X_i(\lambda)$ satisfying the equation

$$A(\lambda) X_i = p_{iX} i.$$

In what follows we shall denote by $B(\lambda, p_i) = \|b_{kj}(\lambda, p_i)\|$ $(k, j = 1, 2, \dots, n)$ the matrix $\|A(\lambda) - p_{iE}\|$, and by $\omega(\lambda, p_i)$ its determinant, i.e.

$$\omega(\lambda, p_i) = \mathrm{Det}\, B(\lambda, p_i) = 0. \tag{1}$$

By $C(\lambda, p_i) = \|c_{kj}(\lambda, p_i)\|$ $(k, j = 1, 2, \dots, n)$, where $c_{kj}(\lambda, p_i)$ is the cofactor of the element $b_{jk}(\lambda, p_i)$ in the determinant $\omega(\lambda, p_i)$, we shall denote the adjugate matrix for the matrix $B(\lambda, p_i)$.

Suppose that for some prescribed value of the parameter λ from the interval $\lambda_0 \le \lambda \le \lambda^*$, say $\lambda = \lambda_0$, the value of the eigenvalue $p_i(\lambda)$ is known to us:

$$\text{for } \lambda = \lambda_0 \qquad p_i(\lambda) = p_i^{(0)}. \tag{2}$$

Suppose, moreover, that:

1) all the functions $a_{kj}(\lambda)$ are defined and continuous on the whole interval $\lambda_0 \leq \lambda \leq \lambda^*$ and have continuous derivatives on this interval;

2) the trace of the matrix $C(\lambda, p_i)\dfrac{\partial B(\lambda, p_i)}{\partial p_i}$,

$$\mathrm{Sp}\left[C(\lambda, p_i)\frac{\partial B(\lambda, p_i)}{\partial p_i}\right]$$

at the point $(\lambda_0, p_i^{(0)})$ is different from zero.

It is required to find approximate values of the eigenvalue $p_i(\lambda)$ for prescribed values of the parameter $\lambda > \lambda_0$.

For this purpose we proceed as follows. Differentiating equation (1) with respect to λ and then solving it with respect to $dp_i/d\lambda$, we obtain

$$\frac{dp_i}{d\lambda} = -\frac{\partial\omega(\lambda, p_i)/\partial\lambda}{\partial\omega(\lambda, p_i)/\partial p_i}.$$

By virtue of the lemma proved in ([1]), the last equation can be rewritten in the form (cf. ([2]))

$$\frac{dp_i}{d\lambda} = -\frac{\mathrm{Sp}\left[C(\lambda, p_i)\,\frac{\partial B(\lambda, p_i)}{\partial\lambda}\right]}{\mathrm{Sp}\left[C(\lambda, p_i)\,\frac{\partial B(\lambda, p_i)}{\partial p_i}\right]}. \tag{3}$$

Let the trace of the matrix

$$C(\lambda, p_i)\,\frac{\partial B(\lambda, p_i)}{\partial p_i}$$

be different from zero at all points of the domain $G^{(i)}$ of variation of λ and p_i, containing the point $(\lambda_0, p_i^{(0)})$, i.e.

$$\mathrm{Sp}\left[C(\lambda, p_i)\,\frac{\partial B(\lambda, p_i)}{\partial p_i}\right] \neq 0 \quad \text{in } G^{(i)}. \tag{4}$$

Then, in order to determine, for prescribed λ, the solutions of equation (1), or, what is the same thing, the eigenvalue values $p_i(\lambda)$ of the matrix $A(\lambda)$, we numerically integrate equation (3) on the interval $\lambda_0 \leq \lambda \leq \lambda^*$ with the initial condition (2).

When computing the elements of the adjugate matrix $C(\lambda, p_i)$, we proceed as follows. Represent the matrix $B(\lambda, p_i)$ in the form

$$B(\lambda, p_i) = \left\| \begin{matrix} P(\lambda, p_i) & u(\lambda, p_i) \\ v(\lambda, p_i) & b_{n,n}(\lambda, p_i) \end{matrix} \right\|,$$

where $P(\lambda, p_i)$ is a matrix of order $n-1$ with determinant $\overline{\Delta}(\lambda, p_i)$ different from zero in the domain $G^{(i)}$;

$$v(\lambda, p_i) = \{b_{n,1}(\lambda, p_i), \ldots, b_{n,n-1}(\lambda, p_i)\}, \qquad u(\lambda, p_i) = \begin{pmatrix} b_{1,n}(\lambda, p_i) \\ \cdot \\ \cdot \\ b_{n-1,n}(\lambda, p_i) \end{pmatrix}.$$

In the case when the determinant $\overline{\Delta}(\lambda, p_i)$ vanishes at some point of the domain $G^{(i)}$, the matter is reduced to the case under consideration by a corresponding interchange of rows and columns of the matrix $B(\lambda, p_i)$.

Proceeding now analogously to the way we did in ([1]), the adjugate matrix $C(\lambda, p_i)$ can be represented in the form

$$C(\lambda, p_i) = \overline{\Delta}(\lambda, p_i)\, C^*(\lambda, p_i),$$

where

$$C^*(\lambda, p_i) = \left\| \begin{matrix} F(\lambda, p_i) & -P^{-1}(\lambda, p_i) u(\lambda, p_i) \\ -v(\lambda, p_i) P^{-1}(\lambda, p_i) & 1 \end{matrix} \right\|,$$

$$F(\lambda, p_i) = P^{-1}(\lambda, p_i) u(\lambda, p_i) v(\lambda, p_i) P^{-1}(\lambda, p_i).$$

The computation of the values of the elements of the inverse matrix $P^{-1}(\lambda, p_i)$ may be carried out, for example, by the method of variation of the parameter ([3]). For this it is necessary additionally to assume that, under condition (2), the inverse matrix $P^{-1}(\lambda, p_i)$ is known to us:

$$\text{for } \lambda = \lambda_0 \qquad p_i(\lambda) = p_i^{(0)}, \qquad P^{-1}(\lambda, p_i) = P_0^{-1}. \tag{5}$$

Thus, the entire computational process reduces to the numerical integration, on the interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$, of the system of two equations

$$\frac{dp_i}{d\lambda} = -\frac{\operatorname{Sp}\left[C^*(\lambda, p_i)\dfrac{\partial B(\lambda, p_i)}{\partial \lambda}\right]}{\operatorname{Sp}\left[C^*(\lambda, p_i)\dfrac{\partial B(\lambda, p_i)}{\partial p_i}\right]}, \tag{6}$$

$$\frac{dP^{-1}(\lambda, p_i)}{d\lambda} = -P^{-1}(\lambda, p_i)\frac{dP(\lambda, p_i)}{d\lambda}P^{-1}(\lambda, p_i).$$

with the initial conditions (5).

It is easy to show that in the domain $G^{(i)}$ each column of the matrix $C^*(\lambda, p_i)$ consists of components of the eigenvector X_i belonging to the eigenvalue $p_i(\lambda)$.

Above we assumed that condition (4) is satisfied. Let us note that if all eigenvalues $p_1(\lambda), p_2(\lambda), \dots, p_n(\lambda)$ of the matrix $A(\lambda)$ take distinct values on the interval $\lambda_0 \leqslant \lambda \leqslant \lambda^*$, then condition (4) will be satisfied on this interval for any $i = 1, 2, \dots, n$.

Let us also note that of particular interest are the cases when, at some point of the domain $G^{(i)}$ which is a solution of equation (1), the trace of the matrix

$$C(\lambda, p_i)\frac{\partial B(\lambda, p_i)}{\partial p_i}$$

vanishes. In these cases we proceed analogously to how we did in the corresponding cases in (4).

2°. The proposed method is also applicable to the computation of eigenvalues and eigenvectors of constant matrices. In this case the constant matrix D is represented as the sum of two matrices D_0 and D_1 in such a way that the eigenvalues of the matrix D_0 and the matrix P_0^{-1} for the matrix $\|D_0 - pE\|$ are easily determined. Then the matrix $D_\lambda = D_0 + \lambda D_1$ for $\lambda = 1$ coincides with the original matrix D, while for $\lambda = 0$ its eigenvalues and the matrix P_0^{-1} are known.

We treat the matrix D_λ analogously to what was set out in 1°, and integrate the obtained system of differential equations of the form (6) on the interval $0 \leqslant \lambda \leqslant 1$. The desired values of the eigenvalues and the components of the eigenvectors are obtained at $\lambda = 1$.

3°. **Example.** Suppose it is required to find the eigenvalues and the corresponding eigenvectors of the matrix

$$A(\lambda) = \begin{Vmatrix} 0 & \lambda & \lambda & 2\lambda - 1 \\ \lambda & -\lambda & \lambda & \lambda \\ \lambda & \lambda - 1 & \lambda & \lambda \\ 2\lambda - 1 & \lambda & \lambda & -1 \end{Vmatrix} \tag{7}$$

for the following values of the parameter λ: 0; 0.05; 0.1; ... ; 0.95; 1.

For $\lambda = 0$ the eigenvalues of matrix (7) take the values: $p_1^{(0)} = 0$; $p_2^{(0)} = 0.618034$; $p_3^{(0)} = -1$; $p_4^{(0)} = -1.618034$.

Choosing the step of numerical integration of the system (6) equal to 0.05 up to $\lambda = 0.25$, and then equal to 0.025, and using the Runge-Kutta method,

we obtain for the eigenvalue $p_1(\lambda)$ of matrix (7) the following results

λ	p_1	p_1 exact	λ	p_1	p_1 exact
0.05	−0.0444889	−0.0444888	0.55	−0.522644	−0.522640
0.10	0.0760228	−0.0760220	0.60	−0.610978	−0.610973
0.15	0.0931888	−0.0931867	0.65	−0.699806	−0.699801
0.20	−0.100207	−0.100204	0.70	−0.788633	−0.788627
0.25	−0.111198	−0.111196	0.75	−0.877083	−0.877076
0.30	−0.143182	−0.143179	0.80	−0.964834	−0.964826
0.35	−0.199032	−0.199029	0.85	−1.051585	−1.051576
0.40	−0.270642	−0.270639	0.90	−1.137040	−1.137027
0.45	−0.350819	−0.350816	0.95	−1.220882	−1.220868
0.50	−0.435529	−0.435525	1.00	−1.302789	−1.302775

For the components of the corresponding eigenvector X_1 at $\lambda = 1$ we obtain the values: −2.302858; 1.000055; 1.000060; 1. The residual vector: −0.000023; 0.000008; 0.000005; 0.000046.

Note added in proof. After the present note had been submitted for publication, the author became aware of a paper by A. A. Dorodnitsyn (DAN, **126**, No. 6, 1959), in which differential equations were also obtained for determining the eigenvalues and eigenvectors of the symmetric matrix $C = A + \varepsilon B$ (A, B constant matrices, ε a parameter).

Received
22 X 1959

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.