# MATHEMATICS

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## Abstract

## Full Text

*MATHEMATICS*

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# INTERPOLATION AND QUADRATURE FORMULAS ON THE CLASSES $W_s^\alpha$ AND $E_s^\alpha$

*(Presented by Academician A. M. Kolmogorov, 20 XII 1959)*

Let $W_s^\alpha(1)$ (and, respectively, $E_s^\alpha(1)$) denote the class of complex-valued functions $f(x_1, \dots, x_s) = f(\mathbf{x})$ having period 1 in each variable and expandable in the Fourier series

$$f(\mathbf{x}) = \sum_{\mathbf{m}} C_{\mathbf{m}} e^{2\pi i(\mathbf{m},\mathbf{x})}, \tag{1}$$

where

$$\sum_{\mathbf{m}} (\overline{m}_1 \dots \overline{m}_s)^\alpha |C_{\mathbf{m}}|^2 \leqslant 1 \quad \left(\text{respectively } |C_{\mathbf{m}}| \leqslant \frac{1}{(\overline{m}_1 \dots \overline{m}_s)^\alpha}\right). \tag{2}$$

Here $\overline{m} = \max(1, |m|)$. For the classes $W_s^\alpha$ and $E_s^\alpha$, see the works ($^{1,2}$).

In the present paper upper and lower estimates are given for the quantities

$$\Delta_1(\alpha) = \min_{\mathbf{a}} \inf_{\varphi_k(\mathbf{x}) \in L_2} \sup_{f(\mathbf{x}) \in W_s^\alpha(1)} \int_0^1 \cdots \int \left| f(\mathbf{x}) - \sum_{k=1}^N f\left(\frac{k\mathbf{a}}{N}\right) \varphi_k(\mathbf{x}) \right|^2 d\mathbf{x} \tag{3}$$

$$\Delta_2(\alpha) = \min_{\mathbf{a}} \inf_{\varphi_k(\mathbf{x}) \in L_2} \sup_{f(\mathbf{x}) \in E_s^\alpha(1)} \int_0^1 \cdots \int \left| f(\mathbf{x}) - \sum_{k=1}^N f\left(\frac{k\mathbf{a}}{N}\right) \varphi_k(\mathbf{x}) \right|^2 d\mathbf{x}.$$

From $W_s^\alpha(1) \subset E_s^\alpha(1)$ it follows at once that $\Delta_1(\alpha) \leqslant \Delta_2(\alpha)$. Conversely, from $f(\mathbf{x}) \in E_s^\alpha(1)$ it follows that, for any $\varepsilon > 0$, $\frac{1}{C(\varepsilon)} f(\mathbf{x}) \in W_s^{\alpha-1/2-\varepsilon}(1)$, whence $\Delta_2(\alpha) \leqslant C^2(\varepsilon)\Delta_1(\alpha - 1/2 - \varepsilon)$. Using this inequality, one can obtain estimates for $\Delta_2(\alpha)$ from estimates for $\Delta_1(\alpha)$; however, by this method only less complete results are obtained for $\Delta_2(\alpha)$ (see Theorem 2).

**Theorem 1.**

$$\frac{1}{2N^\alpha} \leqslant \Delta_1(\alpha) \leqslant C(\alpha, s)\frac{\ln^{\alpha(2s-1)} N}{N^\alpha} \quad \text{for } \alpha \geqslant 1,\ s \geqslant 2. \tag{4}$$

The upper estimate holds for $N$ prime.

**Proof.** Let $\varphi_k(\mathbf{x})$ be expanded in the Fourier series convergent in the mean,

$$\varphi_k(\mathbf{x}) = \sum_{\mathbf{n}} C_{\mathbf{n},k} e^{2\pi i(\mathbf{n},\mathbf{x})},$$

and let the series (1) converge to $f(\mathbf{x})$ at the points

$$\mathbf{x} = \vec{\xi}_k = \frac{k\mathbf{a}}{N} \quad (k = 1, 2, \dots, N;\ \mathbf{a} \text{ is an integer vector}).$$

Then, using the notation $\delta_{\mathbf{mn}} = 0$ for $\mathbf{m} \neq \mathbf{n}$, $\delta_{\mathbf{mm}} = 1$, one can show that

$$\int_0^1 \cdots \int \left| f(\mathbf{x}) - \sum_{k=1}^N f(\vec{\xi}_k)\varphi_k(\mathbf{x}) \right|^2 d\mathbf{x} = \sum_{\mathbf{n}} \left| \sum_{\mathbf{m}} C_{\mathbf{m}} \left\{ \sum_{k=1}^N C_{\mathbf{n}k} e^{2\pi i(\mathbf{m},\vec{\xi}_k)} - \delta_{\mathbf{mn}} \right\} \right|^2 = \sum_{\mathbf{n}} \left| \sum_{\mathbf{m}} C_{\mathbf{m}} \lambda_{\mathbf{mn}} \right|^2. \tag{5}$$

Let us note that, for any fixed $\mathbf{n}$,

$$\Delta_1(\alpha) \geq \min_a \inf_{\varphi_k(\mathbf{x}) \in L_2} \sup_{f(\mathbf{x}) \in W_s^\alpha(1)} \left| \sum_m C_m \lambda_{\mathbf{mn}} \right|^2 = \min_a \inf_{\varphi_k(\mathbf{x}) \in L_2} \sum_m \frac{|\lambda_{\mathbf{mn}}|^2}{(\bar{m}_1 \dots \bar{m}_s)^{2\alpha}}. \tag{6}$$

But

$$\begin{aligned}\sum_m \frac{|\lambda_{\mathbf{mn}}|^2}{(\bar{m}_1 \dots \bar{m}_s)^{2\alpha}} = \sum_{k,l=1}^N C_{n,k} \bar{C}_{n,l} &\left( \sum_m \frac{e^{\frac{2\pi i(k-l)(\mathbf{m},\mathbf{a})}{N}}}{(\bar{m}_1 \dots \bar{m}_s)^{2\alpha}} \right) \\ &- \sum_{k=1}^N \frac{C_{n,k} e^{\frac{2\pi i k(\mathbf{n},\mathbf{a})}{N}} + \partial_{n,k} e^{-\frac{2\pi i k(\mathbf{n},\mathbf{a})}{N}}}{(\bar{n}_1 \dots \bar{n}_s)^{2\alpha}} + \frac{1}{(\bar{n}_1 \dots \bar{n}_s)^{2\alpha}}.\end{aligned}$$

Denoting by $R_n(\mathbf{a})$ the minimum of the written expression with respect to $C_{n1}, \dots, C_{nN}$, after transformations we obtain

$$R_n(\mathbf{a}) = (\bar{n}_1 \dots \bar{n}_s)^{-2\alpha} - \frac{(\bar{n}_1 \dots \bar{n}_s)^{-4\alpha}}{\sum\limits_{(\mathbf{m},\mathbf{a}) \equiv (\mathbf{n},\mathbf{a}) \,(\bmod N)} (\bar{m}_1 \dots \bar{m}_s)^{-2\alpha}} = \frac{1}{(\bar{n}_1 \dots \bar{n}_s)^{2\alpha} + \left( \sum\limits_{\substack{(\mathbf{m},\mathbf{a}) \equiv (\mathbf{n},\mathbf{a}) \\ \mathbf{m} \neq \mathbf{n}}} (\bar{m}_1 \dots \bar{m}_s)^{-2\alpha} \right)^{-1}}. \tag{7}$$

Therefore

$$R_n(\mathbf{a}) \geq \frac{1}{(\bar{n}_1 \dots \bar{n}_s)^{2\alpha} + (\bar{m}_1 \dots \bar{m}_s)^{2\alpha}} \tag{8}$$

for any $\mathbf{m} \neq \mathbf{n}$, $(\mathbf{m}, \mathbf{a}) \equiv (\mathbf{n}, \mathbf{a}) \pmod N$. Using Dirichlet' s principle, for $s \geq 2$ one can prove that for every $\mathbf{a}$ there exist $\mathbf{m}$ and $\mathbf{n}$ with these properties, and moreover $\bar{m}_1 \dots \bar{m}_s \leq \sqrt{N}$, $\bar{n}_1 \dots \bar{n}_s \leq \sqrt{N}$. Then the first of inequalities (4) will follow from (6) and (8).

To obtain the upper estimate in (4), let us note that, by virtue of (2),

$$\sum_n \left| \sum_m C_m \lambda_{\mathbf{mn}} \right|^2 \leq \sum_n \sum_m \frac{|\lambda_{\mathbf{mn}}|^2}{(\bar{m}_1 \dots \bar{m}_s)^{2\alpha}}. \tag{9}$$

Therefore, by virtue of (5), and also in view of the fact that the inner sum in (9) depends only on $C_{n1}, \dots, C_{nN}$,

$$\Delta_1(\alpha) \leq \min_a \inf_{\varphi_k(\mathbf{x}) \in L_2} \sum_n \sum_m \frac{|\lambda_{\mathbf{mn}}|^2}{(\bar{m}_1 \dots \bar{m}_s)^{2\alpha}} = \min_a \sum_n \inf_{C_{nk}} \sum_m \frac{|\lambda_{\mathbf{mn}}|^2}{(\bar{m}_1 \dots \bar{m}_s)^{2\alpha}} = \min_a \sum_n R_n(\mathbf{a}). \tag{10}$$

Using (7), we may write

$$\begin{aligned} \Delta_1(\alpha) &\leq \min_a \sum_n \left\{ (\bar{n}_1 \dots \bar{n}_s)^{-2\alpha} - \frac{(\bar{n}_1 \dots \bar{n}_s)^{-4\alpha}}{\sum\limits_{(\mathbf{m},\mathbf{a}) \equiv (\mathbf{n},\mathbf{a})} (\bar{m}_1 \dots \bar{m}_s)^{-2\alpha}} \right\} \\ &= \min_a \sum_{\mu=0}^{N-1} \left\{ \sum_{(\mathbf{n},\mathbf{a}) \equiv \mu} (\bar{n}_1 \dots \bar{n}_s)^{-2\alpha} - \frac{\sum\limits_{(\mathbf{n},\mathbf{a}) \equiv \mu} (\bar{n}_1 \dots \bar{n}_s)^{-4\alpha}}{\sum\limits_{(\mathbf{n},\mathbf{a}) \equiv \mu} (\bar{n}_1 \dots \bar{n}_s)^{-2\alpha}} \right\}. \end{aligned} \tag{11}$$

From the trivial inequality

$$u_1 + u_2 + \ldots - \frac{u_1^2 + u_2^2 + \ldots}{u_1 + u_2 + \ldots} \le 2(u_2 + u_3 + \ldots),$$

valid for $u_1 \ge u_2 \ge \cdots \ge 0$, and from (11) we obtain

$$\Delta_1(\alpha) \le 2 \min_a \sum_{\mu=0}^{N-1} \sum_{(\mathbf{n},\mathbf{a})\equiv\mu}{}' (\bar{n}_1 \ldots \bar{n}_s)^{-2\alpha}, \tag{12}$$

where the prime means that the largest term in the sum has been omitted (or one of the largest, if there are several). Put $\varepsilon(\mathbf{a};\mathbf{n}) = 0$ if the term $(\bar{n}_1 \ldots \bar{n}_s)^{-2\alpha}$ is omitted in (12), and $\varepsilon(\mathbf{a};\mathbf{n}) = 1$ if it is not omitted. Put also $\delta_N(z) = 1$ if $z \equiv 0 \pmod N$, and $\delta_N(z) = 0$ otherwise. Finally, for convenience of notation, introduce the notation $\bar{n}_1 \ldots \bar{n}_s = |\mathbf{n}|$, and write $\mathbf{m} < \mathbf{n}$ if $\mathbf{m} \ne \mathbf{n}$ and $|\mathbf{m}| \le |\mathbf{n}|$. Then, from the rule for omitting terms in (12), it follows that $\varepsilon(\mathbf{a};\mathbf{n}) \le \sum_{m<\mathbf{n}} \delta_N((\mathbf{m}-\mathbf{n},\mathbf{a}))$, and inequality (12) may be continued as follows:

$$\begin{aligned}
\Delta_1(\alpha) &\le 2 \min_a \sum_n \frac{\varepsilon(\mathbf{a};\mathbf{n})}{|\mathbf{n}|^{2\alpha}} \le 2 \min_a \sum_{|\mathbf{n}|<N/2} \frac{\varepsilon(\mathbf{a};\mathbf{n})}{|\mathbf{n}|^{2\alpha}} + 2 \sum_{|\mathbf{n}|\ge N/2} \frac{1}{|\mathbf{n}|^{2\alpha}} \\
&\le 2 \left( \min_a \sum_{|\mathbf{n}|<N/2} \frac{\varepsilon(\mathbf{a};\mathbf{n})}{|\mathbf{n}|^2} \right)^{\alpha} + O\left( \frac{\ln^{s-1} N}{N^{2\alpha-1}} \right) \\
&\le 2 \left( \min_a \sum_{|\mathbf{n}|<N/2} |\mathbf{n}|^{-2} \sum_{m<\mathbf{n}} \delta_N((\mathbf{m}-\mathbf{n},\mathbf{a}))^{\alpha} \right) + O\left( \frac{\ln^{s-1} N}{N^{2\alpha-1}} \right) \\
&= 2 \left( \min_a \sum_{\substack{\mathbf{m}<\mathbf{n} \\ |\mathbf{n}|<N/2}} \frac{\delta_N((\mathbf{m}-\mathbf{n},\mathbf{a}))}{|\mathbf{n}|^2} \right)^{\alpha} + O\left( \frac{\ln^{s-1} N}{N^{2\alpha-1}} \right).
\end{aligned} \tag{13}$$

Replacing in (13) $\min_a$ by the average over all possible integer vectors $\mathbf{a}$, and using the fact that, for prime $N$, from $\mathbf{m} < \mathbf{n}$, $|\mathbf{n}| < N/2$ it follows that $\mathbf{m} \not\equiv \mathbf{n} \pmod N$, we have

$$\begin{aligned}
\Delta_1(\alpha) &\le 2\left[\frac{1}{N^s}\sum_{a_1,\dots,a_s=0}^{N-1}\sum_{\substack{\mathbf{m}<\mathbf{n}\\|\mathbf{n}|<N/2}}\frac{\delta_N((\mathbf{m}-\mathbf{n},\mathbf{a}))}{|\mathbf{n}|^2}\right]^{\alpha} + O\left(\frac{\ln^{s-1}N}{N^{2\alpha-1}}\right)\\
&= 2\left[\frac{1}{N^s}\sum_{\substack{\mathbf{m}<\mathbf{n}\\|\mathbf{n}|<N/2}}|\mathbf{n}|^{-2}\sum_{a_1,\dots,a_s=0}^{N-1}\delta_N((\mathbf{m}-\mathbf{n},\mathbf{a}))\right]^{\alpha} + O\left(\frac{\ln^{s-1}N}{N^{2\alpha-1}}\right)\\
&= 2\left[\frac{1}{N}\sum_{\substack{\mathbf{m}<\mathbf{n}\\|\mathbf{n}|<N/2}}|\mathbf{n}|^{-2}\right]^{\alpha} + O\left(\frac{\ln^{s-1}N}{N^{2\alpha-1}}\right)\\
&\le 2\left[\frac{1}{N}\sum_{|\mathbf{n}|<N/2}\frac{1}{|\mathbf{n}|^2}\sum_{|\mathbf{m}|\le|\mathbf{n}|}1\right]^{\alpha} + O\left(\frac{\ln^{s-1}N}{N^{2\alpha-1}}\right)\\
&= 2\left[\frac{1}{N}\sum_{|\mathbf{n}|<N/2}\frac{1}{|\mathbf{n}|^2}O(|\mathbf{n}|\ln^{s-1}|\mathbf{n}|)\right]^{\alpha} + O\left(\frac{\ln^{s-1}N}{N^{2\alpha-1}}\right) = O\left(\frac{\ln^{\alpha(2s-1)}N}{N^{\alpha}}\right),
\end{aligned}$$

which proves the theorem. From the theorem just proved it immediately follows that

$$\frac{1}{2N^{\alpha}} \le \Delta_2(\alpha) \le C(\alpha,s,\varepsilon)\,\frac{\ln^{(\alpha-1/2)(2s-1)}N}{N^{\alpha-1/2-\varepsilon}}, \qquad \varepsilon>0, \quad \alpha\ge\frac{3}{2}+\varepsilon.$$

However, one can obtain a sharper result by observing that from $f(\mathbf{x})\in E_s^{\alpha+1/2}(1)$ it follows that

$$\sum_{m}{}' \left|\frac{(\bar m_1\dots\bar m_s)^{\alpha}C_m}{\sqrt{\ln(\bar m_1+3)\dots\ln(\bar m_s+3)}\;\ln\ln(\bar m_1+3)\dots\ln\ln(\bar m_s+3)}\right|^2 \le C_0^2(s),$$

and, considering the class of functions $W_s^{\alpha}(C_0(s))$, defined by the last inequality. Defining for it $\Delta_1'(\alpha)$ by an equality analogous to (3), and carrying out the estimates in the same way as in the preceding theorem, one can obtain

$$\Delta_1'(\alpha) \leqslant O\left(\frac{\ln^{(2s-1)(\alpha+1)+1/2}N\cdot\ln^{2s}\ln N}{N^{\alpha}}\right), \qquad \alpha\geqslant 1, \quad s\geqslant 2.$$

Hence, since $\Delta_2(\alpha)\leqslant\Delta_1'(\alpha-\frac{1}{2})$, we obtain

**Theorem 2*.**

$$\frac{1}{2N^\alpha} \leqslant \Delta_2(\alpha) \leqslant C(\alpha,s)\frac{\ln^{(2s-1)\alpha+s} N\cdot\ln^{2s}\ln N}{N^{\alpha-1/2}}, \qquad \text{for } \alpha \geqslant \tfrac{3}{2}, \quad s \geqslant 2.$$

The theorems proved make it possible, using the parallelepipedal grids introduced by N. M. Korobov, to interpolate values of functions from the classes $W_s^\alpha$ and $E_s^\alpha$ with greater accuracy than is allowed by ordinary interpolation formulas with a uniform grid of interpolation nodes, which in our case can give a mean-square error of order $N^{-2\alpha/s}$.

From the theorems proved there follows, in particular, the possibility of constructing quadrature formulas for integration with an arbitrary square-summable weight

$$\int_0^1 \cdots \int f(x)\rho(x)\,dx \approx \sum_{k=1}^N p_k(\rho) f\left(\frac{ka}{N}\right) \tag{14}$$

with a sufficiently good remainder term. Indeed, if, for example, $f(x) \in W_s^\alpha(1)$ and $\rho(x) \in L_2$, and

$$\int_0^1 \cdots \int \left| f(x) - \sum_{k=1}^N f\left(\frac{ka}{N}\right)\varphi_k(x)\right|^2 dx \leqslant O\left(\frac{\ln^{(2s-1)\alpha} N}{N^\alpha}\right),$$

then

$$\left|\int_0^1 \cdots \int f(x)\rho(x)\,dx - \sum_{k=1}^N f\left(\frac{ka}{N}\right)\int_0^1 \cdots \int \varphi_k(x)\rho(x)\,dx\right| \leqslant O\left(\frac{\ln^{(s-1/2)\alpha} N}{N^{\alpha/2}}\right),$$

i.e. this quadrature formula can no longer be substantially improved, since the following general theorem holds:

**Theorem 3.** Let $\rho(x) \in W_s^\beta(1)$, with $\beta \leqslant \alpha$, $\alpha+\beta \geqslant 1$, $s \geqslant 2$. Then there exist numbers $p_k(\rho)$ and an integer vector $a$ such that, for every $f(x) \in W_s^\alpha(1)$,

$$\left|\int_0^1 \cdots \int f(x)\rho(x)\,dx - \sum_{k=1}^N p_k(\rho) f\left(\frac{ka}{N}\right)\right| \leqslant C(\alpha,\beta,s)\frac{\ln^{(\alpha+\beta)(s-1/2)} N}{N^{(\alpha+\beta)/2}}. \tag{15}$$

On the other hand, there exists a function $\rho(x) \in W_s^\beta(1)$ such that, whatever $a$ and $p_k(\rho)$ may be, there will always be an $f(x) \in W_s^\alpha(1)$ such that the left-hand side of (15) is greater than $\frac{1}{2}N^{-(\alpha+\beta)/2}$.

The proof is carried out analogously to the proof of Theorem 1.

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## References

1. N. M. Korobov, *DAN*, **124**, No. 6, 1207 (1959).
2. S. A. Smolyak, *DAN*, **131**, No. 1 (1960).
3. V. S. Ryabenkii, *DAN*, **131**, No. 5 (1960).

* After the completion of this work, it became known to the author that V. S. Ryabenkii had independently obtained a somewhat more accurate upper estimate for $\Delta_2(\alpha)$ (see (9)).

*Note: Figure translations are in progress. See original paper for figures.*