A. Arhangel' skii and A. Taimanov

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Abstract

Full Text

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On a Theorem of V. Ponomarev

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In paper ([1]) V. Ponomarev proved for the first time that, under sufficiently broad assumptions, an open mapping $f : X \to Y$ of a normal space X onto a normal space Y extends to an open mapping $\varphi : \beta X \to \beta Y$ of the Čech extension βX onto the Čech extension βY. In doing so, V. Ponomarev obtained his result as a special case of a corresponding theorem for strongly continuous multivalued mappings. However, in V. Ponomarev' s theorem it was assumed that the mapping f is bicompact, i.e. that the inverse images $f^{-1}y$ (and in the case of multivalued mappings also the images fx) of all points $y \in Y$ (respectively $x \in X$) are bicompact.

One of the authors of the present note (A. Taimanov) freed (in the case of single-valued mappings) V. Ponomarev's theorem from the condition of bicompactness. After this A. Arhangel' skii gave a considerably simpler proof of V. Ponomarev' s theorem strengthened in this way, extending it again to multivalued mappings.

Theorem 1. Let f be a continuous, simultaneously closed and open single-valued mapping of a normal space X onto a normal space Y. Then the unique continuous extension $\varphi : \beta X \to \beta Y$ of the mapping f is an open mapping of the space βX onto βY.

Theorem 2. Let f be a strongly continuous, simultaneously closed and open Y-bicompact mapping of a normal space X onto a normal space Y. Then the unique strongly continuous extension* $\varphi : \beta X \to \beta Y$ of the mapping f is an open mapping of the space βX onto βY.

Theorem 1 is contained in Theorem 2, to whose proof we now turn.

Notation. x is an arbitrary point of the space βX; U is an arbitrary neighborhood of the point x in βX; V is such a neighborhood of the point x in βX that $[V] \subset U$. All closures are denoted by square brackets and are considered in βX, respectively in βY.

Lemma. The set $\varphi x \cap [Y \setminus f(X \cap U)]$ is empty.

Suppose the contrary; then there exists a point $y \in \varphi x$ contained in $[Y \setminus f(X \cap U)]$.

The sets

$$B_0 = f(X \cap [V]) \subseteq f(X \cap U) \subseteq Y,$$

$$B_1 = Y \setminus f(X \cap U) \subseteq Y$$

are closed in Y; the first because the mapping f is closed, the second because this mapping is open. Obviously, $B_0 \cap B_1$ is empty. Since Y is normal, it follows that $[B_0] \cap [B_1]$ is empty.

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* Existing under our assumptions by virtue of the results of V. Ponomarev ([2]) (corollary of Theorem 2 on p. 270). In the present note we use the notation and terminology of papers ([1],[2]) (in particular, Y-bicompactness of the mapping f means that all $f(x)$ are bicompact).

By our assumption, $y \in [B_1]$. On the other hand, $x \in [X \cap V] \subseteq [X \cap [V]]$, whence, by the strong and hence, in particular, oblique continuity of the mapping φ, it follows that

$$y \in \varphi x \subseteq [\varphi(X \cap [V])] = [f(X \cap [V])] = [B_0],$$

i.e. $y \in [B_0] \cap [B_1]$. The contradiction obtained proves the lemma.

We prove Theorem 2, i.e. we prove that every point $y \in \varphi x$ is an interior point of the set φU. Since $[V] \subset U$, it is enough to prove that y is an interior point of the set $\varphi[V]$. By the lemma, $y \notin [Y \setminus f(X \cap V)]$. The set $\varphi[V] \subset \beta Y$ is bicompact; hence $\beta Y \setminus \varphi[V]$ is open in βY. If y is not an interior point of the set $\varphi[V]$, then in every neighborhood of it there are points of the set $\beta Y \setminus \varphi[V]$ and, consequently, of the set $Y \setminus \varphi[V]$. But then

$$y \in [Y \setminus \varphi[V]] \subseteq [Y \setminus \varphi V] \subseteq [Y \setminus f(X \cap V)],$$

which contradicts the lemma. The theorem is proved.

Moscow State University
named after M. V. Lomonosov

Institute of Mathematics
Siberian Branch of the Academy of Sciences of the USSR

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REFERENCES

1. V. Ponomarev, DAN, **126**, No. 4, 716 (1959).

2. V. Ponomarev, DAN, **124**, No. 2, 268 (1959).

Note: Figure translations are in progress. See original paper for figures.