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Abstract

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MATHEMATICS

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ON CRITICAL POINTS OF PERIODIC FUNCTIONALS

(Presented by Academician P. S. Aleksandrov on 18 XI 1959)

1. The theorems on coverings of spheres and on the category of projective space, formulated by L. A. Lyusternik and L. G. Shnirelman in 1929 ⁽¹⁾, have since then been reproved many times by various authors. A natural generalization of the theorem on the category of projective space is the replacement of the special mapping—the central symmetry of the sphere S^n —by a more general mapping of the sphere into itself.

We shall formulate one of the results obtained along this path by M. A. Krasnosel' skii ⁽²⁾.

Let A be a periodic transformation of the unit sphere S^n onto itself, satisfying two conditions: the transformations A, A^2, \dots, A^{p-1} have no fixed points on the sphere S^n , and $A^p = I$ (I is the identity transformation). Let Π_n be the topological manifold obtained from the sphere S^n by identifying sets of points $x, Ax, \dots, A^{p-1}x$. Then the category of the space Π_n is equal to $n + 1$.

We note that the result formulated by M. A. Krasnosel' skii is a generalization of a theorem of A. I. Fet ⁽³⁾.

In the present paper we consider an application of M. A. Krasnosel' skii' s theorem to estimating the number of critical, relative to a transformation A , functionals on the sphere of a Banach space. The analogous problem for the case when A is a linear periodic operator was considered by Yu. G. Borisovich ⁽⁴⁾.

Let E be a regular Banach space with uniformly differentiable norm, and let A be a periodic operator mapping the unit sphere S of the space E onto itself.

A closed compact set \mathcal{E} is called a set of the first kind ^(5,6) if $A\mathcal{E} = \mathcal{E}$ and if in each of its connected components there is no pair of points of the form $x, A^i x$, where $1 \leq i \leq p - 1$. A compact set \mathcal{E} has genus n if it can be enclosed in the

union of n sets of the first kind and is not contained in any union of $n - 1$ such sets.

Let

$$E_{m_1} \subset E_{m_2} \subset \dots \subset E_{m_i} \subset \dots$$

be a sequence of finite-dimensional subspaces of increasing dimension of the space E , invariant under the operator A , and moreover

$$P_{m_i} Ax = AP_{m_i} x \quad (i = 1, 2, \dots), \quad x \in S,$$

where P_{m_i} ($i = 1, 2, \dots$) are the operators of orthogonal projection onto the corresponding subspaces E_{m_i} .

From M. A. Krasnosel'skiĭ's theorem the following follows.

Theorem 1. *On the unit sphere S of the space E there exist sets of every genus.*

2. Let on the sphere S there be defined a weakly continuous and uniformly differentiable functional $F(x)$, satisfying the conditions:
 - 1) $F(Ax) = F(x)$, $x \in S$;
 - 2) $F(x) > 0$, $x \neq \theta$, $F(\theta) = 0$;
 - 3) $(\Gamma x, x) \neq 0$ for $x \neq \theta$ and $\Gamma\theta = \theta_1$, where by Γ is denoted the gradient operator of the functional $F(x)$.

Suppose, moreover, that on the sphere S the operator A satisfies the Lipschitz condition

$$\|Ax - Ay\| \leq k\|x - y\|, \quad x, y \in S.$$

A point $x_0 \in S$ is called a critical point of the functional $F(x)$, belonging to its critical value c , if for some real $\lambda \neq 0$

$$\Gamma x_0 = \lambda L_\alpha x_0,$$

where $F(x_0) = c$ and $L_\alpha x = \text{grad } \|x\|^\alpha$, $\alpha > 1$.

Let M_{m_i} ($i = 1, 2, \dots$) be the class of compact closed sets on the sphere S , whose genus is not less than m_i . The numbers

$$c_{m_i} = \sup_{E \in M_{m_i}} \inf_{x \in E} F(x) \quad (i = 1, 2, \dots)$$

are called the critical numbers of the functional $F(x)$.

Theorem 2. *The functional $F(x)$, satisfying conditions 1)–3), has at least a countable number of distinct critical numbers c_{m_i} .*

For each set $\mathcal{E} \subset M_{m_i}$ one can construct a continuous operator $\chi_E(x)$, mapping the sphere S into itself and such that

$$A\chi_E(x) = \chi_E(Ax), \quad x \in S,$$

where

$$F[\chi_E(x)] \geq F(x) + \frac{a(x)}{4}(L_\alpha^{-1}Dx, Dx), \quad x \in S,$$

where $a(x)$ is some continuous and positive functional on S ; L_α^{-1} is the operator inverse to the operator L_α , and the operator Dx is defined by the formula

$$Dx = \Gamma x - \frac{(\Gamma x, x)}{\alpha} L_\alpha x.$$

By the scheme proposed by M. A. Krasnosel'skii (7), the following basic theorem can be proved.

Theorem 3. *The functional $F(x)$, satisfying conditions 1)–3), has on the sphere S at least a countable number of distinct critical points.*

For functionals satisfying the conditions of Theorem 3, one may pose the problem of studying the structure of the set of critical points when the corresponding critical points of the functional $F(x)$ coincide.

This question is completely characterized by Theorem 4.

Theorem 4. *If $c_{m_k} = c_{m_{k+1}} = \dots = c_{m_{k+l}} = c$, then on the sphere S there exists a set of critical points belonging to the critical value c , whose genus is not less than $l + 1$.*

Theorem 3 is a direct generalization of analogous theorems for functionals of a more particular form (4, 8).

3. Under some additional assumptions concerning the smoothness of the operators A , L_α , L_α^{-1} , and Γ , Theorem 3 can be proved by the method of orthogonal trajectories (8, 9).

We introduce the operators

$$\Gamma_1 x = \Gamma x - \frac{(\Gamma x, x)}{(L_\alpha x, x)} L_\alpha x, \quad x \in S,$$

$$\bar{\Gamma}_1 x = L_\alpha^{-1} \Gamma_1 x - \frac{(L_\alpha^{-1} \Gamma_1 x, L_\alpha x)}{(L_\alpha x, x)} x, \quad x \in S,$$

$$Dx = \bar{\Gamma}_1 x + \sum_{i=1}^{p-1} \bar{B}_{A^i x} \bar{\Gamma}_1 A^i x, \quad x \in S,$$

where

$$\bar{B}_{A^i x} = B_{A^{p-1}x} B_{A^{p-2}x} \cdots B_{A^i x} \quad (i = 1, 2, \dots, p-1).$$

Here $B_{A^i x}$ denotes the Fréchet derivative of the operator A at the point $A^i x$.

Consider the differential equation

$$\frac{dx(t)}{dt} = Dx(t), \quad (1)$$

where $x(t)$ is the image, differentiable with respect to the numerical parameter t , in E of the segment $[0, 1]$. If the operator D satisfies a Lipschitz condition in some domain $Q \supset S$, then for some $t_0 > 0$ on the segment $[0, t_0]$ the differential equation (1) has a unique solution $x(t)$ which at $t = 0$ becomes the prescribed element $x_0 \in S$. This solution is continuous in t and depends continuously on the initial condition. The solution $x(t, x_0)$ of equation (1) will be called the trajectory issuing from the point $x_0 \in S$.

Define on the sphere S the operator

$$x(x_0, t) = x(t, x_0), \quad x_0 \in S, \quad t \in [0, t_0].$$

Theorem 5. For every $t \in [0, t_0]$ and every $x_0 \in S$,

$$x(x_0, t) \in S,$$

and moreover

$$x(Ax_0, t) = Ax(x_0, t).$$

Theorem 6. Let $x(x, t)$, $t \in [0, t_0]$, be the trajectory issuing from an arbitrary point $x \in S$. Then

$$F[x(x, t_0)] - F(x) \geq N \left[\sum_{i=0}^{p-1} (L_\alpha^{-1} \Gamma_1 A^i x, \Gamma_1 A^i x) \right]^2, \quad (2)$$

where $N > 0$ is some constant.

For the case of a Hilbert space, formula (2) was indicated by M. A. Krasnosel'skii. Relation (2), showing that along a trajectory the functional $F(x)$ increases, makes it possible to prove Theorem 3.

The construction of orthogonal trajectories can be carried out in an analogous way for any sufficiently smooth surface $\Phi(x) = a$, where the functional $\Phi(x)$ satisfies the condition $\Phi(Ax) = \Phi(x)$.

The theorems formulated on critical points of a periodic functional can be applied to the study of certain classes of systems of nonlinear integral equations.

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Note: Figure translations are in progress. See original paper for figures.

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