Mathematics

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Abstract

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ON SOME INTERPOLATION PROBLEMS IN THE CLASS H_p FOR $p < 1$

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1. In papers ([1–5]), among other questions, the following problem is considered: given n points $\lambda_1, \lambda_2, \ldots, \lambda_n$ ($|\lambda_k| < 1$) and n numbers $c_1, c_2, \ldots, c_n$, it is required to find a function of the class H_p, interpolating at the points λ_k the values c_k, with the least norm* in comparison with other functions of the same class interpolating the same values. The function sought in the problem will henceforth be called minimal. In the cited papers the existence, general form, and uniqueness of the minimal function are proved in the case $p \geq 1$. In particular, in ([4]) it is proved that, in order that the function $f(z)$ be minimal in the class H_p, $p \geq 1$, with respect to the given λ_k and c_k, it is necessary and sufficient that $f(z)$ have the form

$$f(z) = A \prod{}' \frac{z - \alpha_k}{1 - \overline{\alpha}_k z} \prod_{k=1}^{n-1} (1 - \overline{\alpha}_k z)^{2/p} \prod_{k=1}^{n} (1 - \overline{\lambda}_k z)^{-2/p}, \tag{1}$$

where $A, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ ($|\alpha_k| \leq 1$) are certain constants, and the product $\prod'$ extends over some or all α_k for which $|\alpha_k| < 1$. The existence of a minimal function in the class H_p, $p > 0$, follows trivially from the compactness principle.

The author of the present article considers the form of the minimal function and the question of its uniqueness in the case $p < 1$.

2. Theorem 1. If the function $f(z)$ is minimal in the class H_p, $p > 0$, with respect to the given λ_k and c_k ($k = 1, 2, \ldots, n$); $b(z)$ is the Blaschke product formed for the zeros of $f(z)$, then the function

$$g(z) = \left[\frac{f(z)}{b(z)}\right]^{p'/p} b(z), \qquad \text{where } p' > p, \tag{2}$$

is minimal and, moreover, the unique minimal function in the class $H_{p'}$ with respect to λ_k and to its values (for multiple λ_k, with respect to the values of the corresponding derivatives) at these points.

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* The norm of a function $f(z) \in H_p$, $p > 0$, is defined by the equality

$$\|f\|_{H_p} = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p \, d\theta \right\}^{1/p}, \qquad \text{where } f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}).$$

Obviously, for $p < 1$ the name "norm" can be retained only conditionally, since in this case the triangle inequality is not satisfied in general.

The scheme of the proof is as follows. Suppose that there exists a minimal function $g^*(z) \in H_{p'}$, $g^*(z) \neq g(z)$, interpolating the same values as $g(z)$ at the points λ_k (i.e. $\|g^*\|_{H_{p'}} \leqslant \|g\|_{H_{p'}}$); then we would obtain:

1) if $\|g^*\|_{H_{p'}} < \|g\|_{H_{p'}}$, then it follows that $\|f^*\|_{H_p} < \|f\|_{H_p}$, where

$$f^*(z) = \left[\frac{g(z)}{b(z)}\right]^{p'/p-1} g^*(z)$$

interpolates the same values as $f(z)$ at the points λ_k, which is impossible, since $f(z)$ is minimal;

2) if $\|g^*\|_{H_{p'}} = \|g\|_{H_{p'}}$, then it follows that almost everywhere $\left|g^*(e^{i\theta})\right| = \left|g(e^{i\theta})\right|$; hence it follows that $g^*(z) \equiv g(z)$, which is impossible by assumption.

Theorem 2. A minimal function in the class H_p, $0 < p < 1$, has the form (1).

For the proof it is enough to apply Theorem 1 for $p' \geqslant 1$. Then the minimal function (2), as is known, must have the form (1) (where $p = p'$), and therefore

$$f(z) = \left[\frac{g(z)}{b(z)}\right]^{p'/p} b(z) \tag{3}$$

will have the form (1), since the transformation (3) changes only p in the expression (1).

Definition. We shall call the *index of minimality* (p_0) the exact lower bound of those p for which the expression (1) (with fixed α_k, λ_k and zeros) is a minimal function with respect to the points λ_k and its values at these points, and the *index of uniqueness* (p_0^*) the exact lower bound of those p for which the indicated expression is the unique minimal function.

Obviously, by definition $p_0 \leqslant p_0^*$.

From Theorem 1 and from the investigation of the interpolation problem in the case $\lambda_1 = \lambda_2 = 0$; $c_1 = 1$, $c_2 > 0$, it follows:

Theorem 3. The index of minimality p_0 is equal to the index of uniqueness and, with a suitable choice of λ_k and c_k ($k = 1, 2, \ldots, n$), can assume any values from the interval $[0,1]$; for $p = p_0$ the function (1) is minimal, but may be a nonunique minimal function.

Example. The function $f_1(z) = (1+z)^2$ is minimal in the class H_1 in the case $\lambda_1 = \lambda_2 = 0$ with respect to the values $c_1 = 1,\ c_2 = 2$ (i.e. $f_1(0) = 1,\ f_1'(0) = 2$), but the function $f_p(z) = (1+z)^{2/p}$ will not be minimal in any class H_p for $p < 1$.

Remark. For $\lambda_1 = \lambda_2 = 0$ and any $p < 1$, there exists a unique value of the ratio $|c_2/c_1|$ for which in the class H_p there exist two distinct minimal functions with respect to $\lambda_1, \lambda_2;\ c_1, c_2$ (i.e. interpolating the same values).

3. From the investigation of the interpolation problem in the case $\lambda_1 = \lambda_2 = 0$, one obtains an exact estimate for $|f'(z)|$. In particular, for $f(z) \in H_p,\ p < 1$, the estimate

$$|f'(z)| \leqslant \frac{h(z)\|f\|_{H_p}}{(1-|z|^2)^{1/p+1}}, \tag{4}$$

holds, where $h(z)$ is a bounded real function defined by the equality

$$h(z) = \max\left\{2\left(1-\frac{p}{2}\right)^{1/p}\left(1+\frac{|z|}{\sqrt{p(2-p)}}\right),\right.$$

$$\left.\frac{2}{p}\,\frac{\rho(z) - \dfrac{|z|}{2-p} + \sqrt{\left(\dfrac{|z|}{2-p}\right)^2 + \dfrac{p}{2-p}}}{\left[1 + 2\left(\dfrac{|z|}{2-p}\right)^2 + \dfrac{p}{2-p} - \dfrac{2|z|}{2-p}\sqrt{\left(\dfrac{|z|}{2-p}\right)^2 + \dfrac{p}{2-p}}\right]^{1/p}}\right\}.$$

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Note: Figure translations are in progress. See original paper for figures.