V. S. ZHURAVSKII

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Abstract

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V. S. ZHURAVSKII

ON THE QUESTION OF THE GROUP OF ABELIAN EXTENSIONS OF ABELIAN GROUPS

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In the present paper we consider the construction of the group $\mathrm{Ext}(F, T)$ of abelian extensions of an abelian group T by means of an abelian group F.

In proving the theorems we shall use the following assertion:

Let A, B, and C be abelian groups such that the sequence

$$0 \to A \to B \to C \to 0$$

is exact. Then, for any abelian group G, the sequences

$$0 \to \mathrm{Hom}(G, A) \to \mathrm{Hom}(G, B) \to \mathrm{Hom}(G, C) \to \mathrm{Ext}(G, A) \to \\ \to \mathrm{Ext}(G, B) \to \mathrm{Ext}(G, C) \to 0; \quad (1)$$

$$0 \to \mathrm{Hom}(C, G) \to \mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G) \to \\ \to \mathrm{Ext}(C, G) \to \mathrm{Ext}(B, G) \to \mathrm{Ext}(A, G) \to 0. \quad (2)$$

Theorem 1. Let F be a torsion-free abelian group of finite rank r; let $f_1, f_2, \dots, f_r$ be a system of linearly independent elements of the group F, and denote $\sum\{f_i\} = C$, $i = 1, 2, \dots, r$; finally, let T be a periodic abelian group. Then the group $\mathrm{Ext}(F, T)$ is isomorphic to the quotient group of the group $\mathrm{Ext}(F/C, T)$ by its periodic part V:

$$\mathrm{Ext}(F, T) \cong \mathrm{Ext}(F/C, T)/V.$$

Indeed, since $\mathrm{Hom}(C, T) \cong T \times r$ and $\mathrm{Ext}(C, T) = 0$, the application of the exact sequence (2) to the group T and to the exact sequence

$$0 \to C \xrightarrow{\alpha} F \xrightarrow{\beta} F/C \to 0$$

gives the exact sequence

$$T \times r \xrightarrow{w} \operatorname{Ext}(F/C, T) \xrightarrow{\beta^*} \operatorname{Ext}(F, T) \to 0.$$

The group $T \times r$ is periodic, the homomorphism w maps it into the periodic part V of the group $\operatorname{Ext}(F/C, T)$, and since β^* is an epimorphism and $\operatorname{Ext}(F, T)$ is a torsion-free group, the homomorphism w maps $T \times r$ onto the group V, which is equivalent to the assertion of Theorem 1.

Corollary. Let T be a periodic abelian group and let H be a torsion-free abelian group of rank one, containing an element h, in whose characteristic

$$\chi(h) = (k_1, k_2, \dots, k_i, \dots)$$

all numbers k_i are finite. Then $\operatorname{Ext}(H, T)$ is isomorphic to the quotient group of the complete direct sum $U = \sum^*_{p_i \in P} T_{p_i}/p_i^{k_i} T_{p_i}$ by its periodic part

$$V = \sum_{p_i \in P} T_{p_i}/p_i^{k_i} T_{p_i}, \quad \text{where } T_{p_i} \text{ is the } p_i\text{-primary component of the group } T$$

with respect to the prime number p_i, and p_i runs through the entire set P of prime numbers,

$$\operatorname{Ext}(H, T) \cong \left(\sum^*_{p_i \in P} T_{p_i}/p_i^{k_i} T_{p_i} \right) \Big/ \left(\dot{\sum}_{p_i \in P} T_{p_i}/p_i^{k_i} T_{p_i} \right).$$

Thus, denoting by C the cyclic subgroup of the group H generated by the element h, we obtain that

$$H/C = \dot{\sum}_{p_i \in P} C(p_i^{k_i}),$$

where $C(p_i^{k_i})$ is the cyclic group of order $p_i^{k_i}$. Then

$$\operatorname{Ext}(H/C, T) \cong \sum^*_{p_i \in P} \operatorname{Ext}(C(p_i^{k_i}), T) \cong \sum^*_{p_i \in P} T/p_i^{k_i} T \cong \sum^*_{p_i \in P} T_{p_i}/p_i^{k_i} T_{p_i}.$$

The periodic part V of the group $\operatorname{Ext}(H/C, T)$ is isomorphic to

$$\dot{\sum}_{p_i \in P} T_{p_i}/p_i^{k_i} T_{p_i}.$$

In particular, if F is a complete torsion-free group, then a stronger theorem holds.

Theorem 2. Let F be a complete torsion-free abelian group of rank r (r a finite or infinite cardinal number), and let T be any reduced abelian group. Then

$$\mathrm{Ext}(F,T) \cong \sum_r^{*}(\mathrm{Ext}(R/Z,T)/T),$$

where R is the additive group of all rational numbers, Z is its subgroup of all integers, and $\sum_r^*$ denotes the complete direct sum of r copies.

Indeed, $F = R \times r$, $\mathrm{Hom}(R,T) = 0$, $\mathrm{Hom}(Z,T) \cong T$, and $\mathrm{Ext}(Z,T) = 0$. For the exact sequence

$$0 \to Z \to R \to R/Z \to 0,$$

by (2) we obtain the exact sequence

$$0 \to T \xrightarrow{\alpha} \mathrm{Ext}(R/Z,T) \to \mathrm{Ext}(R,T) \to 0.$$

Identifying the group T with its image $T\alpha$, we obtain

$$\mathrm{Ext}(R,T) \cong \mathrm{Ext}(R/Z,T)/T;$$

therefore

$$\mathrm{Ext}(F,T) = \mathrm{Ext}(R \times r,T) \cong \sum_r^{*}\mathrm{Ext}(R,T) \cong \sum_r^{*}(\mathrm{Ext}(R/Z,T)/T).$$

Lemma. Let F and T be torsion-free abelian groups, and suppose that the type of every element $f \neq 0$ of F is greater than the type of every element $t \neq 0$ of T. Then $\mathrm{Hom}(F,T) = 0$.

Let $\alpha \in \mathrm{Hom}(F,T)$, $f \in F$, and $f\alpha = t \in T$. Comparing the types of the elements f and t for any $f \in F$, we see that $\alpha = 0$.

Theorem 3. Let the torsion-free abelian groups F and T satisfy the conditions:

1) F is a group of finite rank r;
2) the type of every nonzero element of F is greater than the type of every nonzero element of T.

If F_1 is a pure subgroup of rank one of the group F, then

$$\mathrm{Ext}(F,T) = \mathrm{Ext}(F/F_1,T) + \mathrm{Ext}(F_1,T).$$

Proof. According to the lemma, $\mathrm{Hom}(F/F_1, T) = \mathrm{Hom}(F, T) = \mathrm{Hom}(F_1, T) = 0$. Therefore, for the exact sequence

$$0 \to F_1 \to F \to F/F_1 \to 0$$

we obtain the exact sequence

$$0 \to \mathrm{Ext}(F/F_1, T) \to \mathrm{Ext}(F, T) \to \mathrm{Ext}(F_1, T) \to 0.$$

Now the validity of the theorem follows from the fact that, according to Nunke' s theorem, the group $\mathrm{Ext}(F/F_1, T)$ is complete ([2]).

It is known that every group H of rank one is isomorphic to some subgroup of the additive group R of rational numbers; we shall identify H with this subgroup; by Z we denote its subgroup of all integers.

Theorem 4. Let H be a group of rank one, and suppose that in the characteristic

$$\chi(1) = (k_1, k_2, \dots, k_i, \dots)$$

of the number $1 \in H$ all numbers k_i are finite. Further, let T be any torsion-free group. Denote $T' = \bigcap_{p_i \in P} p_i^{k_i} T$. Then

$$\mathrm{Ext}(H, T) \cong (\Sigma^* T/p_i^{k_i} T)/(T/T').$$

Proof. It is easy to verify that $\mathrm{Hom}(H/Z, T) = 0$, $\mathrm{Ext}(Z, T) = 0$, $\mathrm{Hom}(H, T) \cong T'$, $\mathrm{Hom}(Z, T) \cong T$, and

$$\mathrm{Ext}(H/Z, T) \cong \sum_{p_i \in P}^{*} T/p_i^{k_i} T.$$

Then for the exact sequence

$$0 \to Z \to H \to H/Z \to 0$$

there is the exact sequence

$$0 \to T' \to T \to \sum_{p_i \in P}^{*} T/p_i^{k_i} T \xrightarrow{\beta} \mathrm{Ext}(H, T) \to 0. \tag{3}$$

The kernel of the epimorphism β is isomorphic to the group T/T'; therefore sequence (3) may be written as the exact sequence

$$0 \to T/T' \xrightarrow{\alpha} \sum_{p_i \in P}^{*} T/p_i^{k_i}T \xrightarrow{\beta} \mathrm{Ext}(H,T) \to 0.$$

Identifying the kernel of the epimorphism β with the group T/T', we thereby prove Theorem 4.

Remark. It can be shown that the elements of the kernel $(T/T')\alpha$ of the epimorphism β are precisely those elements of the group

$$\sum_{p_i \in P}^{*} T/p_i^{k_i}T,$$

which can be written in the form of a sequence

$$(t + p_1^{k_1}T,\ t + p_2^{k_2}T,\ \dots,\ t + p_i^{k_i}T, \dots) = (\bar{t}, \bar{t}, \dots, \bar{t}, \dots), \qquad t \in T.$$

These elements form a subgroup A of the group $\sum_{p_i \in P}^{*} T/p_i^{k_i}T$, and $(T/T')\alpha = A$.

Without particular difficulty, the following corollaries of Theorem 4 can be proved.

Corollary 1. If the group T contains elements of type smaller than the type of the group H, then $\mathrm{Ext}(H,T)$ is a mixed group.

Corollary 2. Suppose that the type of every element of the group T is not smaller than the type of the group H. Then the group $\mathrm{Ext}(H,T)$ is torsion-free, and

$$\mathrm{Ext}(H,T) \cong \left(\sum_{p_i \in P}^{*} T/p_i^{k_i}T \right) \Big/ \left(\sum_{p_i \in P}^{*} T_i/p_i^{k_i}T_i \right).$$

Corollary 3. Let the groups H and T satisfy the condition of Theorem 4. In this case the group $\mathrm{Ext}(H,T)$ is equal to zero if and only if at least one of the following conditions is fulfilled:

1) in the characteristic $\chi(1) = (k_1, k_2, \dots, k_i, \dots)$ almost all the numbers k_i are equal to 0;

2) the group T is a p-complete group for almost all those primes for which in $\chi(1)$ the numbers $k_i \neq 0$.

It is known that the concept of a basic subgroup of a given primary abelian group, introduced by L. Ya. Kulikov, has proved very fruitful in the theory of abelian groups. A. G. Kurosh posed the question:

Will the construction of the group $\mathrm{Ext}(F, T)$ be simplified if one introduces into consideration a basic subgroup of the group F or T?

Along this path the following results have been obtained.

Theorem 5. Let F be a group primary with respect to the prime p, and let B be its basic subgroup. If the abelian group T satisfies the conditions:

1) $T[p] = 0$;
2) $pT = T$,

then

$$\mathrm{Ext}(F, T) \cong \mathrm{Ext}(F/B, T).$$

Proof. It is easy to see that

$$\mathrm{Hom}(F/B, T) = \mathrm{Hom}(F, T) = \mathrm{Hom}(B, T) = \mathrm{Ext}(B, T) = 0.$$

And then the application of the exact sequence (2) to the group T and the exact sequence

$$0 \to B \to F \to F/B \to 0$$

proves Theorem 5.

Theorem 6. Let T be a periodic abelian group whose primary components T_p do not contain elements of infinite height, and let B be the direct sum of the basic subgroups of the primary components of the group T,

$$B = \sum_{p \in M} B_p,$$

and let F be such a torsion-free group that $pF = F$ for all primes p for which the primary components $T_p \neq 0$. Then the exact sequence

$$0 \to \mathrm{Hom}(F, T/B) \to \mathrm{Ext}(F, B) \to \mathrm{Ext}(F, T) \to 0$$

holds.

To the group F and the exact sequence

$$0 \to B \to T \to T/B \to 0$$

the exact sequence (1) is applied, and it is proved that

$$\mathrm{Hom}(F, B) = \mathrm{Hom}(F, T) = \mathrm{Ext}(F, T/B) = 0.$$

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REFERENCES

[1] A. G. Kurosh, *Theory of Groups*, Moscow–Leningrad, 1953.
[2] R. Baer, *Math. Ann.*, **135**, no. 3, 219 (1958).
[3] S. Mac Lane, Séminaire Dubreil et Pisot, 7 mai 1956.

Note: Figure translations are in progress. See original paper for figures.