MATHEMATICS

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1960

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Abstract

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APPLICATION OF THE SMALL-PARAMETER METHOD TO THE CONSTRUCTION OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH A DELAYED ARGUMENT

(Presented by Academician I. G. Petrovskii, March 11, 1960)

1. A method is proposed for constructing solutions of ordinary differential equations with a delayed argument, based on the use of the delay as a small parameter. These solutions are sought by means of successive approximations; moreover, as the zeroth approximation one takes the solution obtained in the absence of delay, and the subsequent approximations give a collection of terms that tend to zero together with the delay. In this case each k-th approximation differs from the $(k-1)$-st by terms of order k with respect to the delay.

This method, which may prove useful if the delay is sufficiently small, can be applied formally to equations of a rather broad class. Let there be given, for example, the equation

$$\dot{x}(t) = f[x(t-\mu),\, t], \tag{1}$$

where $f(x, t)$ is some continuous function of its arguments, differentiable with respect to x, and μ is the delay (constant or variable). Let $x^{(0)}(t)$ be the solution of this equation for $\mu = 0$, satisfying the initial condition $x(0) = x_0$. If this equation is rewritten in the form

$$\dot{x}(t) = f[x(t),\, t] + \Delta f[x(t),\, x(t-\mu),\, t], \tag{2}$$

where the function $\Delta f[x(t),\, x(t-\mu),\, t] = f[x(t-\mu),\, t] - f[x(t),\, t]$ vanishes for $\mu = 0$, and if we put $x = x^{(0)} + y$, then the equation for y can be written in the form

$$\dot{y}(t) = py(t) + Y[y(t),\, t] + \Delta Y[y(t),\, y(t-\mu),\, t], \tag{3}$$

where

$$p = \left(\frac{\partial f}{\partial x}\right)_{x=x^{(0)}}, \qquad Y(y,t) = f(x^{(0)} + y, t) - f(x^{(0)}, t) - py,$$

$$\Delta Y[y(t),\, y(t-\mu),\, t] = f[x^{(0)}(t-\mu) + y(t-\mu),\, t] - f[x^{(0)}(t) + y(t),\, t].$$

We shall seek the solution of this equation, with zero initial condition and vanishing for $\mu = 0$, by the method of successive approximations in the form in which it was applied by I. G. Malkin (^{1}). Define the k-th approximation $y^{(k)}$, $k \geq 1$, as the solution of the equation

$$\dot{y}^{(k)} = py^{(k)} + Y(y^{(k-1)}, t) + \Delta Y[y^{(k-1)}(t),\, y^{(k-1)}(t-\mu),\, t], \qquad y^{(0)} = 0, \quad (4)$$

with zero initial condition. For any $k = 1, 2, ...$ we thus obtain a linear nonhomogeneous equation with a known right-hand side, depending on t and μ, and $\Delta Y = 0$ for $\mu = 0$. We obtain

$$y^{(k)}(t) = \int_0^t K(t,\tau)\{Y(y^{(k-1)}, \tau) + \Delta Y[y^{(k-1)}(\tau),\, y^{(k-1)}(\tau-\mu),\, \tau]\}\, d\tau, \quad (4^*)$$

where

$$K(t,\tau) = e^{\omega(t)-\omega(\tau)}, \qquad \omega(t) = \int_0^t p\, dt.$$

The function $y^{(k)}$ differs from $y^{(k-1)}$ by terms of k-th order with respect to μ.

In an analogous way one can construct successive approximations also for systems of the form

$$\dot{x}_s(t) = f_s[x_1(t), \dots, x_n(t), x_1(t-\mu_1), \dots, x_n(t-\mu_n), t] \quad (s = 1, \dots, n), \qquad (5)$$

where f_s are continuous functions of their arguments, differentiable with respect to $x_\sigma(t)$, $x_\sigma(t-\mu_\sigma)$, $\sigma = 1, \dots, n$, and $\mu_1, \dots, \mu_n$ are delays (constant or variable).

2. First of all, the question arises of the existence of the indicated approximations $y^{(k)}$, $k = 1, 2, ...$, and of the convergence of this process. In the particular case when we have linear equations with constant coefficients, i.e. equations of the form

$$\dot{x}_s(t) = a_{s1}x_1(t) + \dots + a_{sn}x_n(t) + b_{s1}x_1(t-\mu_1) + \dots + b_{sn}x_n(t-\mu_n) \qquad (6)$$

$$(s = 1, \dots, n),$$

where $a_{s\sigma}, b_{s\sigma}$ are constants, it can be shown that the approximations $y_s^{(k)}$ exist for all $-\infty < t < \infty$ and all k, and the sequences $\{y_s^{(k)}\}$ converge to functions $y_1, \dots, y_n$, defined for all $-\infty < t < \infty$, if the delays $\mu_1, \dots, \mu_n$ do not exceed a certain bound. We obtain a solution $x_s = x_s^{(0)} + y_s$ of equations (6), satisfying the same initial conditions as the solution $x_s^{(0)}$. If all μ_σ are constant, then this solution will be analytic in $\mu_1, \dots, \mu_n$.

For example, for the equation $\dot{x}(t) = ax(t-\mu)$, where a is a constant number, for constant μ and the initial condition $x(0) = x_0$ we obtain a solution in the form $x(t) = x_0 P(t,\mu)e^{at}$, where $P(t,\mu)$ is represented as a double power series (in t and in μ), convergent for any finite t, if $\mu < \bar{\mu} = 1/|a|e$. For $t = 0$ we have $P(0,\mu) = 1$. The convergence of the series for $P(t,\mu)$ is rapid for sufficiently small μ. For example, for $a = -1$, $\mu = 0.1$, we obtain the third approximation in the form

$$x^{(3)}(t) = x_0[1 - 0.11814t + 0.0067529t^2 - 0.0001939t^3]e^{-t}.$$

The differences between the approximations $x^{(1)}, x^{(2)}$ and $x^{(2)}, x^{(3)}$ do not exceed, respectively, 0.0026, 0.00026. For variable μ, convergence of the approximations $x^{(k)}$ is also guaranteed, provided only that $\mu(t) < \bar{\mu}$ for all t.

In general, the indicated results cannot be extended to equations of arbitrary form (1) or (5). It is not always possible to construct an infinite sequence of approximations by the proposed method, and if it is possible, one cannot always guarantee its convergence. However, if the delay is sufficiently small, it is nevertheless possible in this way to construct an approximate solution, determined by the prescribed initial conditions and satisfying the equations under consideration with a sufficiently high degree of accuracy.

Let the function $f(x,t)$ in equation (1) be defined and have bounded derivatives of the 1st and 2nd order with respect to x in a certain domain $D(|x| \le A,\ |t| \le T)$, and let the solution $x^{(0)}(t)$ also be defined for $|t| \le T$. Then one can guarantee that the function $y^{(1)}$ is defined on the interval $-T+\mu \le t \le T$, the function $y^{(2)}$ on $-T+2\mu \le t \le T$, and so on.

Fix an interval $[-N, L]$, where $N \le L \le T$, and choose functions $U(u)$, $V(u)$ such that $U(u) \ge |f(x,t)|$, $U'(u) \ge |f'_x(x,t)|$, $U''(u) \ge |f''_{xx}(x,t)|$, if $u \ge |x|$, $V(u) \ge |Y(y,t)|$, $V'(u) \ge |Y'_y(y,t)|$, if $u \ge |y|$, for all $-N \le t \le L$. Next form the equation

$$u = LM[V(u) + \mu\widetilde{U}(u_0 + u)], \tag{7}$$

where $\widetilde{U}(u) = U(u)U'(u)$, and M, u_0 are upper bounds for the moduli of the functions $K(t,\tau)$ and $x^{(0)}(t)$ as t and τ vary in the interval $[-N, L]$, while $\tilde{\mu}$ is the upper bound of the delay in this interval. This equation determines the quantity u as a function of the parameter μ. If one seeks the solution of this equation by successive approximations, putting $u^{(0)} = 0$ and

$$u^{(k)} = LM\big[V(u^{(k-1)}) + \tilde{\mu}\,\widetilde{U}(u_0 + u^{(k-1)})\big] \quad (k = 1, 2, \dots), \tag{8}$$

then it is not difficult to show that the quantities $u^{(1)}, \dots, u^{(m)}$, $m = E\left(\frac{N}{2\mu} - \frac{1}{2}\right)$, majorize the functions $y^{(1)}, \dots, y^{(m)}$, respectively, in the interval $[-\mu, L]$, and moreover not only $u^{(k)} \geq |y^{(k)}|$, but also $u^{(k)} - u^{(k-1)} \geq |y^{(k)} - y^{(k-1)}|$. The sequence (8) will converge if $\tilde{\mu}$ does not exceed a certain bound μ_0 (see, for example, (2)). It can be shown that the error of the approximations $y^{(1)}, \dots, y^{(m-1)}$ in the interval $[-\mu, L]$, compared with the function strictly satisfying equation (3), is majorized by the error of the approximations $u^{(1)}, \dots, u^{(m-1)}$, respectively, relative to the exact solution of equation (7). For sufficiently small μ, we are able to find an approximate solution of equation (3) with a very high degree of accuracy.

3. Suppose that the function $f(x,t)$ is defined for all t, and that the solution $x^{(0)}(t)$, as well as all approximations $x^{(0)} + y^{(k)}$, $k = 1, 2, \dots$, are periodic or almost-periodic functions. Then we can construct an infinite sequence of functions $y^{(k)}$, $k = 1, 2, \dots$, and majorize them by means of the solution of an equation of the form (7) on the whole t-axis. Consequently, if μ does not exceed a certain bound, then one can guarantee the convergence of the sequence $\{y^{(k)}\}$ and the existence of a periodic or almost-periodic solution of equation (1).

4. Consider the system of quasilinear equations

$$\dot{x}_s(t) = \sum_{\sigma=1}^{n} a_{s\sigma} x_\sigma(t) + b_{s\sigma} x_\sigma(t - \mu) + X_s \quad (s = 1, \dots, n), \tag{9}$$

($a_{s\sigma}, b_{s\sigma}$ are constants), where X_s are nonlinear, twice differentiable functions of $x_\sigma(t), x_\sigma(t-\mu)$, $\sigma = 1, \dots, n$, periodic in t or not explicitly dependent on t, and μ is a constant delay. One can extend the Lyapunov-Poincaré methods to the system (9), taking μ as the small parameter. For example, it can be shown that if the system (9), for $\mu = 0$, possesses an isolated periodic solution, then for sufficiently small μ it also possesses such a solution (for a system in which the functions X_s do not depend explicitly on t, the period of this solution will depend on μ).

Let, for example, the equation be given

$$\dot{x}(t) = x(t) + (-2 + \cos t)x^2(t - \mu). \tag{10}$$

For $\mu = 0$ it has the unique periodic solution $\varphi(t) = 2/\psi(t)$, where $\psi(t) = 4 - (\cos t + \sin t)$. Putting $x = \varphi + y$, we form the equation for y

$$\dot{y} = py + (-2 + \cos t)y^2 + \bar{f}(\varphi + y),$$

where $p = 1 + 2\varphi(t)(-2 + \cos t)$, $\bar{f}(\varphi + y) = (-2 + \cos t)\{[\varphi(t - \mu) + y(t - \mu)]^2 - [\varphi(t) + y(t)]^2\}$. The first approximation y_1 satisfies the equation $\dot{y}_1 = py_1 + \bar{f}(\varphi)$, which has a unique periodic solution of period 2π. For the second approximation y_2 we obtain the analogous equation $\dot{y}_2 = py_2 + (-2 + \cos t)y_1^2 + \bar{f}(\varphi + y_1)$, which also has a unique periodic solution, and so on. If μ does not exceed a certain bound, then this process converges to a periodic solution of equation (10), tending to the solution $\varphi(t)$ as $\mu = 0$.

If the functions X_s in equations (9) additionally contain a small parameter ε, then the Lyapunov–Poincaré methods can be applied by considering

solution curve depending on two parameters, μ and ε. Estimates of the values of μ and ε for which these solutions exist can be obtained with the aid of majorizing equations of the form (7).

In an analogous way one can consider the question of the existence of almost-periodic solutions of system (9).

5. The proposed method makes it possible to obtain solutions of a certain class, determined uniquely by prescribing only the initial values of the unknown functions. The question arises of the possibility of using them to construct solutions of equations with delay that correspond to prescribed initial functions. In the particular case of linear equations with constant coefficients and constant delay μ of the form

$$\dot{x}_s(t) = a_{s1}x_1(t) + \dots + a_{sn}x_n(t) + b_{s1}x_1(t - \mu) + \dots + b_{sn}x_n(t - \mu) \tag{11}$$

$$(s = 1, \dots, n)$$

one can arrive at the following result.

Consider the solution $\psi_1, \dots, \psi_n$ of system (11) corresponding to some initial functions $\varphi_1, \dots, \varphi_n$ prescribed on the interval $t_0 - \mu \le t < t_0$. By means of the known "method of steps" (see, for example, (3)) we obtain successively the functions $\psi_1, \dots, \psi_n$ on each of the intervals $[t_{k-1}, t_k]$, $k = 1, 2, \dots$, $t_k = t_0 + k\mu$. Suppose that for $t = t_k$ these functions take the values $\psi_s(t_k) = \alpha_{sk}$, $s =$

$1, \dots, n$. Let us now construct a solution $\bar{x}_s(t, t_k)$, $s = 1, \dots, n$, of system (11) by the proposed method of successive approximations under the initial conditions $x_s(t_k) = \alpha_{sk}$. It turns out that if μ is small, then the solution $\psi_1, \dots, \psi_n$ is approximated by the solution $\bar{x}_s(t, t_k)$ for $t \geq t_k$ the better the larger t_k is. The following theorem can be proved:

Theorem. Let two positive numbers L, ε be prescribed arbitrarily, of which L is as large as desired and ε as small as desired. Then, if μ does not exceed a certain bound, there will be a time T such that on the interval $[t_*, t_* + L]$, where $t_* \geq T$, the inequalities

$$|\psi_s(t) - \bar{x}_s(t, t_*)| \leq \varepsilon, \quad s = 1, \dots, n,$$

hold, where $\bar{x}_s(t, t_*)$ is the solution of equations (11), analytic with respect to t, μ and found under the initial conditions $x_s(t_*) = \psi_s(t_*)$, $s = 1, \dots, n$.

Thus, the farther the interval $[t_*, t_* + L]$ is from the initial moment, the less, on this interval and for small μ, any solution of equations (11) differs from a solution that is analytic with respect to t and μ.

Consider, for example, the equation $\dot{x}(t) = -x(t - \mu)$ for $\mu = 0,1$, with initial condition $x(0) = 1$, and suppose that the initial function on the interval $-\mu \leq t < 0$ is continuous and equal to $\varphi(t) = 1 + \Delta\varphi(t)$, where $|\Delta\varphi(t)| < \gamma$, $\varphi(0) = 1$. Then it can be shown that the actual solution $\psi(t)$ differs, for all $t \geq t_2$, from the solution $\bar{x}(t, t_2)$ by no more than $(0.9 + 8.2\gamma)10^{-4}$, and from the solution $\bar{x}(t, t_3)$, for all $t \geq t_3$, by no more than $(0.9 + 7.9\gamma)10^{-6}$.

A substantial question is the extent to which this result can be extended to more complicated equations.

Received
3 III 1960

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Note: Figure translations are in progress. See original paper for figures.