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Abstract

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ASYMPTOTIC EXPANSIONS OF THE EIGENVALUES OF A BOUNDARY-VALUE PROBLEM WITH A PARAMETER

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MATHEMATICS

We consider the eigenvalue problem consisting of the system

$$y'(x) + B(x)y(x) = \lambda\Lambda(x)y(x) \tag{1}$$

and the boundary conditions

$$M\left(\frac{1}{\lambda}\right)y(0) + N\left(\frac{1}{\lambda}\right)y(1) = 0. \tag{2}$$

In the present paper, asymptotic formulas will be obtained for the eigenvalues of problem (1), (2), refining the known asymptotic formulas for eigenvalues due to G. D. Birkhoff and J. D. Tamarkin ($^{1-3}$), and also some new formulas of this type will be obtained. The matrix $B(x)$ $(0 \le x \le 1)$ in system (1) has complex entries, equal to zero on the diagonal (the latter does not restrict the generality of the results), and $\Lambda(x)$ $(0 \le x \le 1)$ is a real diagonal matrix with diagonal elements $\nu_i(x)$ $(0 \le x \le 1;\ i = 1, 2, \dots, r)$. Both of them are m times $(m \ge 1)$ continuously differentiable on the segment $[0, 1]$. In the boundary conditions (2)

$$M(z) = \sum_{k=0}^{p_0} M_k z^k, \qquad N(z) = \sum_{k=0}^{p_1} N_k z^k$$

are polynomials in z with matrix coefficients; λ is a complex parameter.

Theorem 1. *If, in addition to the conditions listed above, the matrix $\Lambda(x)$ $(0 \le x \le 1)$ is required to have everywhere distinct diagonal elements, then for system (1) and $|\lambda| > R > 0$, where R is some constant, there exists a fundamental matrix $Y(x, \lambda)$ $(0 \le x \le 1;\ |\lambda| > R)$, admitting the representation*

$$Y(x,\lambda)=\left[E+\frac{\Phi E}{\lambda}+\frac{\Phi^2 E}{\lambda^2}+\dots+\frac{\Phi^{m-1}E}{\lambda^{m-1}}+\frac{H(x,\lambda)}{\lambda^m}\right]\exp\left[\lambda\int_0^x \Lambda(\xi)\,d\xi\right], \tag{3}$$

where E is the identity matrix; $\Phi S(x)$ is a linear operator on the matrix function $S(x)$ $(0 \le x \le 1)$ of the form $\Phi S(x)=\Phi_0 LS(x)$, in which $LS(x)=$

$$=S'(x)+B(x)S(x), \qquad \Phi_0 S(x)=\sigma S(x)-\int_0^x \delta\{B(\xi)\sigma S(\xi)\}\,d\xi,$$

where $X(x)=\sigma S(x)$ denotes the solution of the equation

$$\Lambda(x)X-X\Lambda(x)=S(x),$$

whose diagonal entries are zero, and $\delta S(x)$ denotes the diagonal matrix having the same diagonal elements as $S(x)$. The matrix $H(x,\lambda)$ $(0\le x\le 1,\ |\lambda|>R>0)$ is analytic in λ in each of

regions: 1) $\operatorname{Re}\lambda \leqslant a,\ |\lambda|>R$; 2) $\operatorname{Re}\lambda > a,\ |\lambda|>R$, where a is any real number (fixed for the given matrix $H(x,\lambda)$), and uniformly in $x\in[0,1]$, moreover

$$\max\|H(x,\lambda)\|<\frac{M_1}{1-R/\lambda} \qquad (x\in[0,1],\ |\lambda|>R),$$

where

$$\|H(x,\lambda)\|=\left\{\sum_{i,k=1}^{r}|h_{ik}(x,\lambda)|^2\right\}^{1/2};$$

$h_{ik}(x,\lambda)$ $(i,k=1,2,\dots,r)$ are the elements of the matrix $H(x,\lambda)$; M_1 is some sufficiently large number.

Formula (3) refines the known asymptotic formulas of G. D. Birkhoff, E. S. Pugachev, and others (^{3,4}).

If the general solution $y(x)=Y(x,\lambda)c$ of system (1) is substituted into the boundary conditions (2), then, to determine the eigenvalues of problem (1), (2), we obtain the equation

$$\Delta(\lambda)\equiv\det\left[M\left(\frac{1}{\lambda}\right)Y(0,\lambda)+N\left(\frac{1}{\lambda}\right)Y(1,\lambda)\right]=0. \tag{4}$$

The determinant $\Delta(\lambda)$ admits the representation

$$\Delta(\lambda) = \sum_{s=0}^{p} a_s\left(\frac{1}{\lambda}\right)\exp[\alpha_s\lambda]. \tag{5}$$

Here $p \leqslant 2^r$; α_s $(s = 0, 1, \dots, p)$ are all possible numbers of the form

$$\alpha_s = \sum_{j=1}^{r} \varepsilon_j^{(s)} \int_0^1 \nu_j(\xi)\, d\xi,$$

where $\varepsilon_j^{(s)}$ $(j = 1, 2, \dots, r;\ s = 0, 1, \dots, p)$ take the values 0 or 1, and the numbers α_s $(s = 0, 1, \dots, p)$ are arranged in decreasing order:

$$a_s\left(\frac{1}{\lambda}\right) = \sum_{j=0}^{m-1} \frac{a_s^{(j)}}{\lambda^j} + \frac{a_s^{(m)}(1/\lambda)}{\lambda^m}, \tag{6}$$

where

$$\left|a_s^{(m)}\left(\frac{1}{\lambda}\right)\right| < \frac{M_2}{(1 - R/|\lambda|)^r} \qquad (|\lambda| > R,\ s = 0, 1, \dots, p),$$

M_2 is some constant.

If now we use the fact that, for the zeros $\lambda_n^{(0)}$ of the Dirichlet polynomial

$$\sum_{s=0}^{p} a_s^{(0)} \exp[\alpha_s\lambda]$$

the formula

$$\lambda_n^{(0)} = \frac{2\pi n i}{\alpha_0 - \alpha_p} + \varphi_0(n) \qquad (n = 0, \pm 1, \dots) \quad (^5),$$

holds, where $\varphi_0(n)$ is a bounded complex-valued function of the form

$$\varphi_0(n) = h_0\left(\exp\left[\frac{2\pi(\alpha_1 - \alpha_p)ni}{\alpha_0 - \alpha_p}\right], \dots, \exp\left[\frac{2\pi(\alpha_{p-1} - \alpha_p)ni}{\alpha_0 - \alpha_p}\right]\right);$$

$h_0(\mu_1, \mu_2, \dots, \mu_{p-1})$ is a function of $p - 1$ complex variables, then one can prove the following theorem.

Theorem 2. Let system (1) satisfy the conditions of Theorem 1 and, in addition, let the matrix $\Lambda(x)$ $(0 \leqslant x \leqslant 1)$ be positive, and let the boundary conditions (2) satisfy the condition

$$\det M_0 \neq 0, \qquad \det N_0 \neq 0. \tag{7}$$

Then, for the eigenvalues λ_n $(n = \pm 1, 2, \ldots)$ of problem (1), (2), there is the representation

$$\lambda_n = \frac{2\pi n i}{a_0 - \alpha_p} + \varphi_0(n) + \frac{\varphi_1(n)}{n} + \cdots + \frac{\varphi_{m-1}(n)}{n^{m-1}} + \frac{\psi_m(n)}{n^m} \quad (n = \pm 1, \ldots) \tag{8}$$

Here $\varphi_s(n)$ $(s = 0, 1, \ldots, m-1)$, $\psi_m(n)$ are complex-valued bounded functions of n, and the functions $\varphi_s(n)$ $(s = 0, 1, \ldots, m-1)$ have the form

$$\varphi_s(n) = h_s\left(\exp\left[\frac{2\pi(\alpha_1 - \alpha_p)ni}{\alpha_0 - \alpha_p}\right], \ldots, \exp\left[\frac{2\pi(\alpha_{p-1} - \alpha_p)ni}{\alpha_0 - \alpha_p}\right]\right),$$

where the functions $h_s(\mu_1, \mu_2, \ldots, \mu_{p-1})$ $(s = 0, 1, \ldots, p-1)$ are piecewise analytic on the surface of the torus $(e^{it_1}, e^{it_2}, \ldots, e^{it_{p-1}})$ $(0 \leq t_j < 2\pi;\ j = 1, 2, \ldots, p-1)$; moreover, the functions $\varphi_s(n)$ $(s = 0, 1, \ldots, m-1)$ are the coefficients of the Taylor expansion at the point $z = 0$ of the function $w = w(z)$ satisfying the equation

$$\sum_{k=0}^{m-1}\left(\frac{z}{\dfrac{2\pi i}{\alpha_0 - \alpha_p} + zw(z)}\right)^k \left[a_0^{(k)} \exp[\alpha_0 w(z)]\right.$$

$$\left. + \sum_{s=1}^{p-1} a_s^{(k)} \exp\left[\frac{2\pi n(\alpha_s - \alpha_p)i}{\alpha_0 - \alpha_p}\right] \exp[\alpha_s w(z)] + a_p^{(k)} \exp[\alpha_p w(z)]\right] = 0. \tag{9}$$

Remark. Theorem 2 remains valid also in the case when the diagonal elements of the matrix $\Lambda(x)$ $(0 \leq x \leq 1)$ are not all positive, but in this case condition (7) must be replaced by the regularity condition for the boundary conditions (2) (3).

In order to obtain expansions similar to expansion (8) in the case when $(\det M_0)(\det N_0) = 0$, one must use the following theorem.

Theorem 3. Let $\Lambda(x) > 0$ and

$$\det\left[M\left(\frac{1}{\lambda}\right) + N\left(\frac{1}{\lambda}\right)\exp\left[\lambda\left\{\int_0^1 \Lambda(\xi)\, d\xi\right\}\right]\right] = \sum_{s=0}^{p} \tilde{a}_s\left(\frac{1}{\lambda}\right)\exp[\alpha_s \lambda], \tag{10}$$

where $\tilde{a}_s(z)$ $(s = 0, 1, \dots, p)$ are polynomials in z; the α_s $(s = 0, 1, \dots, p)$ are the same numbers as in formula (5), and let

$$\tilde{a}_0(z) = \tilde{a}_1(z) = \dots = \tilde{a}_{j_1}(z) \equiv 0, \quad \tilde{a}_{j_2}(z) = \tilde{a}_{j_2+1}(z) = \dots = \tilde{a}_p(z) \equiv 0, \tag{11}$$

$$j_1 < j_2 - 1.$$

Then in formula (5)

$$a_0(z) = a_1(z) = \dots = a_{j_1}(z) \equiv 0, \quad a_{j_2}(z) = a_{j_2+1}(z) = \dots = a_p(z) \equiv 0. \tag{12}$$

If the condition $(\det M_0)(\det N_0) = 0$ holds, then the equation $\Delta(\lambda) = 0$ for finding the eigenvalues, by multiplication by λ^h, where h is a suitably chosen positive integer, can, by virtue of Theorem 3, be reduced to the form

$$\sum_{s=j_1+1}^{j_2-1} \lambda^{k_s} b_s\left(\frac{1}{\lambda}\right) \exp[\alpha_s \lambda] = 0, \tag{13}$$

where $\min_s k_s = 0$ $(s = j_1 + 1, \dots, j_2 - 1)$, and if in formula (3) $m > h$, then

$$b_s\left(\frac{1}{\lambda}\right) = b_s^{(0)} + \frac{b_s^{(1)}}{\lambda} + \dots + \frac{b_s^{(m-h)}}{\lambda^{m-h}} + \frac{\overline{b}_s(1/\lambda)}{\lambda^{m-h+1}}, \tag{14}$$

$$b_s^{(0)} \neq 0$$

and $\overline{b}_s\left(\frac{1}{\lambda}\right)$ remains bounded as $\lambda \to \infty$, $s = j_1 + 1, \dots, j_2 - 1$.

To determine the eigenvalues one may use the method applied to determine the zeros of a quasipolynomial in paper (^{5}). This gives the following result:

Theorem 4. Suppose that system (1) satisfies the conditions of Theorem 2, and that the boundary conditions (2) satisfy the conditions $(\det M_0)(\det N_0) = 0$, and suppose that the characteristic equation $\Delta(\lambda) = 0$, after being reduced to the form (13), has $\max_s k_s = h$, $\min_s k_s = 0$ $(s = j_1 + 1, \dots, j_2 - 1)$ and $h < m$, where m is the number of continuous products in the matrices $B(x)$ and $\Lambda(x)$.

Then to each vertex of the broken line $y = \max_s(\alpha_s x + k_s)$ $(s = j_1 + 1, \dots, j_2 - 1)$ there corresponds a sequence of eigenvalues λ_n $(n = 1, 2, \dots)$, admitting the representation

$$\lambda_n = \gamma \ln \tau_n + i\tau_n + \psi(n) +$$

$$+\sum_{0<k_0+k_1\beta_1+\cdots+k_q\beta_q<m-h+1} b_{k_0k_1\dots k_q}(n)\left(\frac{\ln n}{n}\right)^{k_0}\left(\frac{1}{n^{\beta_1}}\right)^{k_1}\cdots\left(\frac{1}{n^{\beta_q}}\right)^{k_q}+O\left(\frac{1}{n^{m-h+1}}\right)$$

$$(n=1,2,\dots),$$

where the numbers $\gamma, \tau_n, \beta_1, \beta_2, \dots, \beta_q, \psi(n)$ have the same meaning as in formula (16) of paper (^{5}), if it is applied to the quasipolynomial

$$\sum_{s=j_1+1}^{j_2-1} \lambda^{k_s} b_s^{(0)} \exp[\alpha_s \lambda],$$

and $b_{k_0k_1\dots k_q}(n)$ are the coefficients of the Taylor expansion of a certain function of $q+1$ variables.

An analogous result will also hold in the case when not all elements of the matrix $\Lambda(x)$ are positive (but preserve their sign on the segment $[0,1]$), and the boundary conditions (2) are irregular.

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Note: Figure translations are in progress. See original paper for figures.