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Abstract

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A GENERALIZED ABEL INTEGRAL EQUATION

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In the present note the equation is solved in closed form

$$c(x)\int_a^x \frac{\varphi(t)\,dt}{(x-t)^\alpha} + d(x)\int_x^b \frac{\varphi(t)\,dt}{(t-x)^\alpha} = f(x), \qquad 0<\alpha<1. \tag{1}$$

The functions $c(x)$, $d(x)$ given on the interval $[a,b]$ will be assumed not to vanish simultaneously and to have derivatives satisfying a Hölder condition. We shall seek the unknown function $\varphi(x)$ in the class of functions

$$\varphi(x) = \frac{\varphi^*(x)}{(x-a)^{1-\alpha-\varepsilon_1}(b-x)^{1-\alpha-\varepsilon_2}}, \tag{1'}$$

where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, and $\varphi^*(x)$ satisfies a Hölder condition on $[a,b]$. The conditions imposed on $f(x)$ will be indicated below. All quantities entering the equation are assumed real.

We shall show that the solution of equation (1), with the aid of the Riemann boundary-value problem for analytic functions ([1]), is reduced to the solution of Abel' s integral equation *

$$\int_a^x \frac{\varphi(t)\,dt}{(x-t)^\alpha} = g(x), \qquad 0<\alpha<1. \tag{2}$$

The solution of the latter is given ([2,3]) under the assumption of the existence of a derivative $g'(x)$ that is continuous and bounded in a neighborhood of $x=a$. We shall be interested in the solution of equation (2) under other conditions on the behavior of $g(x)$ in a neighborhood of $x=a$. We give, without dwelling on the justification, the necessary formulas.

If $g(x)$ belongs to the class of functions $g(x) = g^*(x)/(x-a)^\gamma$, where $\gamma<\alpha$, and $g^*(x)$ has a derivative satisfying a Hölder condition (a sufficient condition), then the solution of equation (2) belongs to the class $\varphi(x) = \varphi^*(x)/(x-a)^\beta$, where

$\beta = \gamma + 1 - \alpha$, and $\varphi^*(x)$ satisfies a Hölder condition with the same exponent as $g^*(x)$. In this case the solution of equation (2) is given by the formula

$$\varphi(x) = \frac{\sin \alpha\pi}{\pi(x-a)} \left[\alpha \int_a^x \frac{g(t)\,dt}{(x-t)^{1-\alpha}} + \int_a^x \frac{(t-a)g'(t)}{(x-t)^{1-\alpha}}\,dt \right]. \tag{3}$$

To reduce equation (1) to the Riemann boundary-value problem we use Carleman' s idea ([4]) of continuation into the complex plane.

* Some authors call this equation generalized, in distinction to Abel' s equation with kernel $1/\sqrt{x-t}$.

Let z be an arbitrary point of the complex plane. Introduce the function

$$\Phi(z) = [(z-a)(b-z)]^{1/2\alpha - 1/2} \int_a^b \frac{\varphi(t)\,dt}{(t-z)^{\alpha}}. \tag{4}$$

By $[(z-a)(b-z)]^{1/2\alpha-1/2}(t-z)^{-\alpha}$ we mean a certain branch of this function in the plane cut along the segment $a \le x \le b$.

It is easy to see that $\Phi(z)$ will be analytic in the entire complex plane with the cut $[a, b]$. We note that for large $|z|$ and for z sufficiently close to a and b, respectively, we shall have the orders

$$\Phi(z) = O\left(\frac{1}{|z|}\right); \tag{5a}$$

$$\Phi(z) = O\left(\frac{1}{|z-a|^{1/2-1/2\alpha}}\right); \tag{5b}$$

$$\Phi(z) = O\left(\frac{1}{|z-b|^{1/2+1/2\alpha}}\right). \tag{5c}$$

Denote by $\Phi^+(x)$ and $\Phi^-(x)$ the limiting values of $\Phi(z)$ as z tends to a certain point $x \in (a, b)$ respectively from above and from below this interval. Write $\Phi(z)$ in the form

$$\Phi(z) = [(z-a)(b-z)]^{1/2\alpha - 1/2} \left[\int_a^x \frac{\varphi(t)\,dt}{(t-z)^{\alpha}} + \int_x^b \frac{\varphi(t)\,dt}{(t-z)^{\alpha}} \right]. \tag{6}$$

Our immediate task is to determine the limiting values $\Phi^{\pm}(x)$ in terms of the values at the point x of the integrals on the right-hand side of (6).

On the upper bank of the cut in a neighborhood of the point x, assign to the numbers $z - a$ and $b - z$ the argument 0; to the number $t - z$ for $t > x$ (from the second integral) the argument 0, and for $t < x$ (from the first integral) the argument $-\pi$. Then we shall have

$$\Phi^+(x) = [(x-a)(b-x)]^{1/2\alpha - 1/2} \left[e^{\alpha\pi i} \int_a^x \frac{\varphi(t)\,dt}{(x-t)^\alpha} + \int_x^b \frac{\varphi(t)\,dt}{(t-x)^\alpha} \right]. \quad (7)$$

Similarly,

$$\Phi^-(x) = -[(x-a)(b-x)]^{1/2\alpha - 1/2} \left[\int_a^x \frac{\varphi(t)\,dt}{(x-t)^\alpha} + e^{\alpha\pi i} \int_x^b \frac{\varphi(t)\,dt}{(t-x)^\alpha} \right]. \quad (8)$$

For a function represented by the integral (4), formulas (7) and (8) are analogues of the known Sokhotski formulas ([1]), p. 35, for Cauchy-type integrals.

From (7) and (8) we find

$$\int_a^x \frac{\varphi(t)\,dt}{(x-t)^\alpha} = \frac{e^{\alpha\pi i}\Phi^+(x) + \Phi^-(x)}{e^{2\alpha\pi i} - 1} [(x-a)(b-x)]^{1/2 - 1/2\alpha}; \quad (9)$$

$$\int_x^b \frac{\varphi(t)\,dt}{(t-x)^\alpha} = -\frac{\Phi^+(x) + e^{\alpha\pi i}\Phi^-(x)}{e^{2\alpha\pi i} - 1} [(x-a)(b-x)]^{1/2 - 1/2\alpha}. \quad (10)$$

Substituting (9) and (10) into equation (1), we obtain the relation

$$\Phi^+(x) = \frac{e^{\alpha\pi i} d(x) - c(x)}{e^{\alpha\pi i} c(x) - d(x)} \Phi^-(x) + \frac{(e^{2\alpha\pi i} - 1) f(x)}{[(x-a)(b-x)]^{1/2 - 1/2\alpha} \left[e^{\alpha\pi i} c(x) - d(x) \right]}. \quad (11)$$

This is the boundary condition of the Riemann problem (1).

For the proof of the equivalence of the integral equation (1) and the boundary-value problem (11) we shall need the following lemma.

Lemma. If the limiting values of the function $\Phi(z)$, analytic in the plane cut along $[a, b]$ and satisfying conditions (5), are related to the function $\varphi(x)$ by equality (9), then $\Phi(z)$ is related to $\varphi(x)$ by formula (4).

It is obvious that the functions $\Phi(z)$ and $\varphi(x)$ related by equality (4) satisfy equality (9). Suppose now that there exists another function $\Phi_1(z)$, different from (4), which also satisfies equality (9). Let $\Phi_2(z) = \Phi(z) - \Phi_1(z)$. The latter satisfies the homogeneous boundary condition

$$\frac{e^{\alpha\pi i}\Phi_2^+(x)+\Phi_2^-(x)}{e^{2\alpha\pi i}-1}\left[(x-a)(b-x)\right]^{1/2-1/2\alpha}=0$$

or

$$\Phi_2^+(x)=-e^{-\alpha\pi i}\Phi_2^-(x).$$

This boundary-value problem has no nontrivial solutions satisfying all conditions (5). Indeed, its only solution satisfying condition (5a) is

$$\frac{c}{(z-a)^{1/2-1/2\alpha}(z-b)^{1/2+1/2\alpha}}.$$

Condition (5c) is, obviously, violated, and therefore $\Phi_2(z)\equiv 0$, i.e. $\Phi_1(z)\equiv\Phi(z)$. The lemma is proved.

If $\varphi(x)$ is a solution of equation (1) satisfying condition (1′), then the function $\Phi(z)$, defined by formula (4) and satisfying conditions (5), as follows from the derivation, must be a solution of the boundary-value problem (11). Conversely, let $\Phi(z)$ be a solution of the boundary-value problem (11) satisfying conditions (5). We determine $\varphi(z)$ from Abel' s equation (9) (or (10)). The latter, as is known, is unconditionally and uniquely solvable. By the lemma, $\Phi(z)$ and $\varphi(x)$ are related by equality (4), and therefore (10) also holds. Multiplying (9) and (10), respectively, by $c(x)$ and $d(x)$ and taking into account the boundary condition (11), we find that $\varphi(x)$ is a solution of equation (1).

Let us establish the connection between the classes of solutions of the homogeneous boundary-value problem (11) satisfying conditions (5) and the corresponding classes of solutions of the homogeneous equation (1). As is known ([1], p. 427), with respect to the behavior of the solution at the endpoint (a or b), the solutions of problem (11) are divided into two classes*: 1) finite and 2) tending to infinity of integrable order. It is easy to establish that they correspond to solutions of equation (1) having, at the endpoint under consideration, respectively, order greater or less than $1/2-1/2\,\alpha$. Thus, unbounded solutions of problem (11) always correspond to unbounded solutions of equation (1), while bounded solutions of the boundary-value problem may correspond both to bounded and to unbounded solutions of the equation. The number of solutions of problem (11) and the number of solutions of equation (1) in the corresponding classes coincide.

The homogeneous equation (1) with constant coefficients $c(x), d(x)$ has no nontrivial solutions, since the homogeneous boundary-value problem (11), as is easy to show, in this case has only the zero solution.

Consider the nonhomogeneous equation (1). Unlike Abel' s equation, where a right-hand side of the form $g(x)=\dfrac{g^*(x)}{(x-a)^\gamma}$, $\gamma<\alpha$, is admissible, for equation

(1), by virtue of the requirements necessary for the solvability of the boundary-value problem (11) in the class of functions satisfying conditions (5), the free term must be taken from a narrower class (having no singularity at the endpoints).

* It is easy to see that the endpoints are not singular.

We shall indicate some sufficient conditions that may be imposed on $f(x)$.

Let $X(z)$ be the canonical function of the boundary-value problem (11), having at the points a and b, respectively, the orders p_1 and p_2, $p_1 \leqslant 1/2 - 1/2\alpha$ and $p_2 \leqslant 1/2 - 1/2\alpha$. Take $f(x) = (x-a)^{n_1}(b-x)^{n_2} f^*(x)$, where $f^*(x)$ has a derivative satisfying a Hölder condition on $[a, b]$, and $n_1 > 1/2 - 1/2\alpha - p_1$ and $n_2 > 1/2 - 1/2\alpha - p_2$. In this case the right-hand side of equation (9) makes it possible to obtain the solution $\varphi(x)$ by formula (3), and it will belong to the class $(1')$.

Let us note one special case. For $c = d = 1$, equation (1) takes the form

$$\int_a^b \frac{\varphi(t)\,dt}{|x-t|^\alpha} = f(x), \qquad 0 < \alpha < 1. \tag{12}$$

The corresponding boundary-value problem (11) is the jump problem

$$\Phi^+(x) - \Phi^-(x) = \frac{e^{\alpha\pi i} + 1}{[(x-a)(b-x)]^{1/2-1/2\alpha}}\, f(x).$$

It has the unique solution

$$\Phi(z) = \frac{e^{\alpha\pi i} + 1}{2\pi i} \int_a^b \frac{f(t)\,dt}{[(t-a)(b-t)]^{1/2-1/2\alpha}(t-z)}.$$

The conditions (5) are fulfilled by virtue of the properties of a Cauchy-type integral at the endpoints (5).

The solution of equation (12) is obtained from the equation

$$\int_a^x \frac{\varphi(t)\,dt}{(x-t)^\alpha} =$$

$$= \frac{1}{2} f(x) - \frac{1}{2\pi} \operatorname{ctg} \frac{\alpha\pi}{2} [(x-a)(b-x)]^{1/2-1/2\alpha} \int_a^b \frac{f(t)}{[(t-a)(b-t)]^{1/2-1/2\alpha}} \frac{dt}{t-x},$$

for which it is sufficient to take $f(x) = (x-a)^{n_1}(b-x)^{n_2} f^*(x)$, where $f^*(x)$ is the same as above, and n_1 and n_2 are greater than $1/2 - 1/2\alpha$.

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Note: Figure translations are in progress. See original paper for figures.