MELTING OF A SEMI-INFINITE BODY IN A PLANE AND AXISYMMETRIC FLOW OF AN INCOMPRESSIBLE GAS

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Abstract

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MELTING OF A SEMI-INFINITE BODY IN A PLANE AND AXISYMMETRIC FLOW OF AN INCOMPRESSIBLE GAS

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Experiments (^{1,2}) show that if, under a steady flow regime, an axisymmetric solid body has begun to melt, then under certain conditions—for example, a body with a high coefficient of thermal conductivity—a steady melting regime also sets in: the front of the melting wave in the neighborhood of the forward critical point moves with constant velocity.

In the present note it is shown that the complete equations—the Navier–Stokes equations and the equation of heat inflow without allowance for dissipation—with the corresponding boundary conditions on the discontinuity surfaces admit an exact solution giving a steady melting regime in the neighborhood of the critical point of a body occupying a half-space in a plane and axisymmetric flow of an incompressible fluid. In the latter case the body may rotate with constant angular velocity ω about the axis of the flow.

The **plane problem** of finding the steady melting regime in the neighborhood of the critical point of a solid heat-conducting body occupying a half-space ($y < 0$), in a flow of an incompressible fluid, reduces to the simultaneous solution of the unsteady Navier–Stokes and heat-inflow equations for the gas near the melting body

$$\frac{d\mathbf{v}}{dt} = -\frac{1}{\rho}\operatorname{grad} p + \nu\Delta\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0, \quad \frac{dT}{dt} = \chi\Delta T \tag{1}$$

$$(y_0(t) < y < \infty, \quad -\infty < t, \quad -\infty < x < \infty),$$

the unsteady Navier–Stokes and heat-inflow equations in the melt film

$$\frac{d\mathbf{v}_1}{dt} = -\frac{1}{\rho_1}\operatorname{grad} p_1 + \nu_1\Delta\mathbf{v}_1, \quad \operatorname{div}\mathbf{v}_1 = 0, \quad \frac{dT_1}{dt} = \chi_1\Delta T_1 \tag{2}$$

$$(y_1(t) < y < y_0(t), \quad -\infty < t, \quad -\infty < x < \infty),$$

and the equation of thermal conductivity in the solid body

$$\partial T_2/\partial t = \chi_2 \Delta T_2 \quad (-\infty < y < y_1(t), \quad -\infty < t, \quad -\infty < x < \infty) \tag{3}$$

with the following additional conditions:

1) at infinity in the fluid and in the body

$$\lim_{y\to\infty} u = \beta x, \quad \lim_{y\to\infty} T = T_\infty, \quad \lim_{y\to-\infty} T_2 = T_{-\infty}; \tag{4}$$

2) on the contact surface of discontinuity fluid–melt (³)

$$y = y_0(t) \quad (-\infty < t,\ -\infty < x < \infty)$$

$$D_0 = v = v_1, \quad u = u_1, \quad \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \mu_1\left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right),$$

$$-p + 2\mu\frac{\partial v}{\partial y} = -p_1 + 2\mu_1\frac{\partial v_1}{\partial y}, \quad T = T_1, \quad \lambda\frac{\partial T}{\partial y} = \lambda_1\frac{\partial T_1}{\partial y}; \tag{5}$$

3) on the melting surface $y = y_1(t)$ $(-\infty < t,\ -\infty < x < \infty)$

$$\rho_1(D_1 - v_1) = \rho_2 D_1, \quad u_1 = 0, \quad T_1 = T_2 = T_\mathrm{m}, \quad \lambda_2\frac{\partial T_2}{\partial y} - \lambda_1\frac{\partial T_1}{\partial y} = \rho_2 D_1 \delta. \tag{6}$$

Here quantities without a subscript refer to the liquid, subscript 1 to the melt film, and subscript 2 to the solid body; $\mathbf{v} = ui + vj$ is the absolute-velocity vector; p is pressure; ρ is density; T is temperature; μ and ν are respectively the coefficients of absolute and kinematic viscosity; λ and χ are respectively the coefficients of thermal conductivity and thermal diffusivity; D_0 and D_1 are the normal velocities of displacement of, respectively, the contact surface and the melting surface; T_∞ and $T_{-\infty}$ are prescribed constant temperatures; T_m is the melting temperature; δ is the latent heat of melting per unit mass of the body. In writing the last equation of conditions (6) we have neglected the work of the pressure forces and the kinetic energy of the melt, since these terms are negligibly small in comparison with the heat fluxes and the latent heat of melting. If it is assumed that $\rho_1 = \rho_2$, then the last relation in conditions (6) will be exact, since in this case $v_1 = 0$.

Since in what follows we shall seek a stationary melting regime, we do not impose initial conditions (the problem without initial conditions).

Proceeding from the form of the boundary conditions (4) and taking into account that the solution giving the stationary melting regime must be a solution of the type of uniformly propagating waves, we shall seek the solution of the problem formulated above in the form:

for the liquid $(0 < \eta < \infty)$

$$u = \beta x \varphi'(\eta), \quad v = -\sqrt{\beta\nu}\,[\varphi(\eta) + \alpha], \quad p = p_0 - \frac{\rho\beta^2}{2}\left[x^2 + \frac{\nu}{\beta}\varphi^2(\eta)\right] - \beta\mu\varphi'(\eta),$$

$$T = T_{\mathrm{m}} + \Delta T\theta(\eta), \quad \alpha = \frac{a}{\sqrt{\beta\nu}}, \quad \Delta T = T_\infty - T_{\mathrm{m}}, \quad \eta = \left(\frac{\beta}{\nu}\right)^{1/2}(y + at); \; (7)$$

for the melt film $(-\eta^* < \eta_1 < 0)$

$$u_1 = \beta_1 x \varphi_1'(\eta_1), \quad v_1 = -\sqrt{\beta_1\nu_1}\,[\varphi_1(\eta_1) + \alpha_1],$$

$$p_0 = p_{01} - \frac{\rho_1\beta_1^2}{2}\left[x^2 + \frac{\nu_1}{\beta_1}\varphi_1^2(\eta_1)\right] - \beta_1\mu_1\varphi_1'(\eta_1), \tag{8}$$

$$T_1 = T_{\mathrm{m}} + \Delta T\theta_1(\eta_1), \quad \alpha_1 = \frac{a}{\sqrt{\beta_1\nu_1}}, \quad \eta_1 = \left(\frac{\beta_1}{\nu_1}\right)^{1/2}(y + at);$$

for the body $(-\infty < \eta_1 < -\eta^*)$

$$T_2 = T_{\mathrm{m}} - \Delta T\theta_2(\eta_1). \tag{9}$$

The laws of motion of the contact surface and of the front of the melting wave for the stationary melting regime will be

$$y_0(t) = -at, \quad y_1(t) = -at - s, \quad D_0 = D_1 = -a, \quad \eta^* = s\sqrt{\beta_1/\nu_1}. \tag{10}$$

Here the functions $\varphi(\eta)$, $\varphi_1(\eta_1)$, $\theta(\eta)$, $\theta_1(\eta_1)$, $\theta_2(\eta_1)$ and the constant a (the velocity of displacement of the melting front and of the contact surface), $\eta^* = s\sqrt{\beta_1/\nu_1}$ (the dimensionless distance between these surfaces), β_1, p_{01} must be found in the process of solving the problem. To this end we substitute expressions (7)–(10) for the unknown quantities into equations (1)–(3) and conditions (4)–(6). Omitting straightforward calculations, we obtain the following nonlinear boundary-value problem for determining the 5 functions φ, φ_1, θ, θ_1, θ_2 and the 4 parameters $\alpha_1 = a/\sqrt{\beta_1\nu_1}$, $s\sqrt{\beta_1/\nu_1}$, β_1, and p_{01}:

$$\varphi''' + 1 = \varphi'^2 - \varphi\varphi'', \quad \varphi_1''' + 1 = \varphi_1'^2 - \varphi_1\varphi_1''; \tag{11}$$

$$\varphi'(\infty) = 1, \quad \varphi(0) = \varphi_1(0) = 0, \quad \varphi'(0) = r_1^{1/2}\varphi_1'(0),$$
$$\varphi''(0) = r_1^{1/4}m^{-1/2}\varphi_1''(0), \quad \varphi_1'(-\eta^*) = 0; \tag{12}$$

$$\theta'' + \sigma\varphi\theta' = 0, \quad \theta_1'' + \sigma_1\varphi_1\theta_1' = 0, \quad \theta_2'' = \sigma_2\alpha_1\theta_2'; \tag{13}$$

$$\theta(\infty) = 1, \quad \theta(0) = \theta_1(0), \quad l_1\theta'(0) = n^{1/2}r_1^{1/4}\theta_1'(0),$$
$$\theta_1(-\eta^*) = \theta_2(-\eta^*) = 0, \quad \theta_2(-\infty) = k; \tag{14}$$

$$r_2\varphi_1(-\eta^*) = -\alpha_1, \quad \theta_2'(-\eta^*) + l_2\theta_1'(-\eta^*) = -r_2\overline{\Delta}\varphi_1(-\eta^*); \tag{15}$$

$$\beta_1 = \beta r_1^{1/2}, \quad p_{01} = p_0 + \beta\varphi'(0)(\mu - \mu_1). \tag{16}$$

Here the following notation has been introduced:

$$\sigma = \frac{\nu}{\chi}, \quad \sigma_1 = \frac{\nu_1}{\chi_1}, \quad \sigma_2 = \frac{\nu_1}{\chi_2}, \quad r_1 = \frac{\rho}{\rho_1}, \quad r_2 = \frac{\rho_1}{\rho_2}, \quad l_1 = \frac{\lambda}{\lambda_1}, \quad l_2 = \frac{\lambda_1}{\lambda_2},$$
$$m = \frac{\mu}{\mu_1}, \quad n = \frac{\nu}{\nu_1}, \quad k = \frac{T_\mathrm{p} - T_{-\infty}}{T_\infty - T_\mathrm{p}}, \quad \overline{\Delta} = \frac{\rho_2\delta\nu_1}{\lambda_2\left(T_\infty - T_\mathrm{p}\right)}. \tag{17}$$

The system of equations (11) and (13) is a nonlinear system of ordinary differential equations of the 12th order with 12 boundary conditions (12) and (14). From the last two equations (15), two parameters, α_1 and η^*, must be determined. Therefore one may expect that the solution of problem (11)–(15), and consequently also of problem (1)–(6), is determined uniquely.

The axisymmetric problem of the melting of a solid body occupying a half-space and rotating with constant angular velocity ω about the axis of an incident flow of incompressible fluid admits an exact solution for the stationary melting regime in the form: for the gas $(0 < \zeta < \infty)$

$$v_r = \beta r\varphi'(\zeta), \quad v_z = -\sqrt{\beta\nu}\,[2\varphi(\zeta) + \alpha], \quad v_\theta = \beta r g(\zeta),$$
$$p = p_0 - \frac{\rho\beta^2}{2}\left[r^2 + 4\frac{\nu}{\beta}\varphi^2(\zeta)\right] - 2\mu\beta\varphi'(\zeta), \quad T = T_\mathrm{p} + \Delta T\theta(\zeta), \tag{18}$$

$$\zeta = \left(\frac{\beta}{\nu}\right)^{1/2}(z+at);$$

for the melt film $(-\zeta^* < \zeta < 0)$

$$\begin{gathered}
v_{r1} = \beta_1 r\varphi_1'(\zeta_1), \qquad v_{z1} = -\sqrt{\beta_1\nu_1}\,[2\varphi_1(\zeta_1)+\alpha_1], \qquad v_{\theta 1} = \omega r g_1(\zeta_1),\\
p_1 = p_{01} - \frac{\rho_1\beta_1^2}{2}\left[r^2 + 4\frac{\nu_1}{\beta_1}\varphi_1^2(\zeta_1)\right] - 2\mu_1\beta_1\varphi_1'(\zeta_1), \qquad (19)\\
T_1 = T_\mathrm{p} + \Delta T\theta_1(\zeta_1), \qquad \zeta_1 = \left(\frac{\beta_1}{\nu_1}\right)(z+at);
\end{gathered}$$

for the body $(-\infty < \zeta_1 < -\zeta_1^*)$

$$T_2 = T_\mathrm{p} - \Delta T\theta_2(\zeta_1). \qquad (20)$$

The laws of motion of the contact surface and of the melting-wave front for the stationary melting regime are

$$z_0(t) = -at, \qquad z_1(t) = -at - s, \qquad D_0 = D_1 = -a. \qquad (21)$$

Here $\mathbf{v} = v_r\mathbf{i} + v_\theta\mathbf{j} + v_z\mathbf{k}$ is the absolute-velocity vector; r is the radial coordinate; z is the axial coordinate; the parameters $\beta_1, p_{01}, a, \zeta^* = s\sqrt{\beta_1/\nu_1}$ – unknowns; the remaining notation coincides with the notation of the plane problem. Substituting the solution (18)–(21) into equations (1)–(3) and conditions (4)–(6), written in a cylindrical coordinate system, we obtain, for determining 7 functions $\varphi, \varphi_1, g, g_1, \theta, \theta_1, \theta_2$ and 4 parameters $\alpha_1, \xi^* = s\sqrt{\beta_1/\nu_1}, \beta_1$, and p_{01}, the nonlinear boundary-value problem

$$\varphi''' + 1 = \varphi'^2 - 2\varphi\varphi'' - g^2, \quad \varphi_1''' + 1 = \varphi_1'^2 - 2\varphi_1\varphi_1'' - \Omega^2 g_1^2, \quad \Omega = \omega/\beta_1; \quad (22)$$

$$\begin{gathered}
\varphi'(\infty) = 1, \quad \varphi(0) = \varphi_1(0) = 0, \quad \varphi'(0) = r_1^{1/2}\varphi_1'(0),\\
\varphi''(0) = r_1^{1/4}m^{-1/2}\varphi_1''(0), \quad \varphi_1'(-\eta^*) = 0; \qquad (23)
\end{gathered}$$

$$g'' + 2\varphi g' - 2\varphi g = 0, \quad g_1'' + 2\varphi_1 g_1' - 2\varphi_1' g_1 = 0; \qquad (24)$$

$$g(\infty) = 0, \quad g(0) = \Omega r_1^{1/2} g_1(0), \quad g'(0) = \Omega r_1^{1/4} m^{-1/2} g_1'(0), \quad g_1(-\xi^*) = 1; \qquad (25)$$

$$\theta'' + 2\sigma\varphi\theta' = 0, \quad \theta_1'' + 2\sigma_1\varphi_1\theta_1' = 0, \quad \theta_2'' = \sigma_2\alpha_1\theta_2'; \tag{26}$$

$$\theta(\infty) = 1, \quad \theta(0) = \theta_1(0), \quad l_1\theta'(0) = n^{1/2}r_1^{1/4}\theta_1'(0),$$
$$\theta_1(-\zeta^*) = \theta_2(-\zeta^*) = 0, \quad \theta_2(-\infty) = k; \tag{27}$$

$$2r_2\varphi_1(-\xi^*) = -\alpha_1, \quad \theta_2'(-\zeta^*) + l_2\theta_1'(-\zeta^*) = -2r_2\Delta\varphi_1(-\xi^*); \tag{28}$$

$$\beta_1 = \beta r_1^{1/2}, \quad p_{01} = p_0 + 2\beta\varphi'(0)(\mu - \mu_1), \quad \Omega = \frac{\omega}{\beta_1} = \frac{\omega}{\beta}r_1^{-1/2}. \tag{29}$$

The notation for the dimensionless parameters in this problem coincides with the notation (17) in the plane problem. The system of equations (22), (24), and (26) is a nonlinear system of ordinary differential equations of the 16th order with 16 boundary conditions (23), (25), and (27). From the last 2 equations (28), the 2 parameters α_1 and ξ^* must be determined. Therefore one may expect that the solution of problem (22)–(28) is determined uniquely.

The actual construction of solutions of the boundary-value problems (11)–(15) and (22)–(28) can be carried out numerically or approximately, by expanding the solution in a series in the parameter $r_1^{1/2}$, which for metallic bodies melting in an air stream is of order 10^{-2}. We note that the general solution of each of the linear equations (13) and (26) is written in quadratures. The exact solution constructed is readily generalized to the case of a compressible gas with variable viscosity and thermal-conductivity coefficients both in the gas and in the melt film. It is also possible to take into account the dependence of the body' s thermal-conductivity coefficient on temperature (4).

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Note: Figure translations are in progress. See original paper for figures.

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