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Abstract

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MATHEMATICS

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A FAMILY OF AXIAL PLANES OF A TWO-DIMENSIONAL SURFACE OF FOUR-DIMENSIONAL PROJECTIVE SPACE AND SOME PROBLEMS CONNECTED WITH THIS FAMILY

(Presented by Academician P. S. Aleksandrov on 2 III 1960)

1. To each point A_0 of a two-dimensional surface (A_0) of four-dimensional projective space we attach a projective frame formed by five analytic points: the point A_0 , the points A_1, A_4 and A_2, A_3 , of which the first two are the points corresponding to the point A_0 under two successive Laplace transformations of the surface (A_0) in one direction, and the other two—in the other direction. The infinitesimal displacements of the vertices of the frame are defined by the equations $dA_i = \omega_i^j A_j$, where the forms ω_i^j satisfy the structure equations of projective space (1): $D\omega_i^j = [\omega_i^k \omega_k^j]$ ($i, j, k = 0, 1, 2, 3, 4$).

With the indicated choice of the vertices of the frame we shall have

$$\begin{aligned}
 \omega^3 = 0, \quad \omega^4 = 0, \quad \omega_1^2 = 0, \quad \omega_2^1 = 0, \quad \omega_1^3 = 0, \quad \omega_2^4 = 0, \\
 \omega_1^4 = a_1^4 \omega^1, \quad \omega_2^0 = a_2^0 \omega^2, \quad \omega_4^3 = a_4^3 \omega^1, \quad \omega_4^0 = a_4^0 \omega^1, \quad \omega_4^2 = a_4^2 \omega^1, \\
 \omega_2^3 = b_2^3 \omega^2, \quad \omega_1^0 = b_1^0 \omega^2, \quad \omega_3^4 = b_3^4 \omega^2, \quad \omega_3^0 = b_3^0 \omega^2, \quad \omega_3^1 = b_3^1 \omega^2, \\
 \omega_4^1 = a_4^1 \omega^1 + b_4^1 \omega^2, \quad \omega_3^2 = a_3^2 \omega^1 + b_3^2 \omega^2.
 \end{aligned} \tag{1}$$

Continuation of these equations gives

$$\begin{aligned}
 da_1^4 + a_1^4(\omega_0^0 - 2\omega_1^1 + \omega_4^4) &= a_{11}^4\omega^1, \\
 da_2^0 + a_2^0(2\omega_0^0 - \omega_1^1 - \omega_2^2) &= a_{21}^0\omega^1, \\
 da_4^1 + a_4^1(\omega_0^0 - \omega_4^4) &= a_{41}^1\omega^1 + (a_{42}^1 - a_4^3b_3^1)\omega^2, \\
 da_4^0 + a_4^0(2\omega_0^0 - \omega_1^1 - \omega_4^4) + (a_4^1b_1^0 + a_4^3b_3^0)\omega^2 &= a_4^1\omega^1, \\
 da_4^2 + a_4^2(\omega_0^0 - \omega_1^1 + \omega_2^2 - \omega_4^4) + (a_4^0 + a_4^3b_3^2)\omega^2 &= a_{41}^2\omega^1, \\
 db_4^1 + b_4^1(\omega_0^0 + \omega_1^1 - \omega_2^2 - \omega_4^4) &= a_{42}^1\omega^1 + b_{42}^1\omega^2
 \end{aligned} \tag{2}$$

and also 6 equations obtained from equations (2) by the replacement $\begin{pmatrix} 1 & 3 & a \\ 2 & 4 & b \end{pmatrix}$.

2. We shall call an **axial plane of the surface** (A_0) at the given point A_0 a two-dimensional plane along which the three-dimensional osculating planes of the lines of the unique conjugate net of the surface (A_0) passing through the point A_0 intersect. Thus, with every surface of four-dimensional projective space there is associated a two-parameter family of axial planes. The axial planes of the surfaces (A_0) and (A_1) at the points A_0 and A_1 are, respectively, the planes

$$\pi_0 = [A_0, a_4^2A_2 + a_4^3A_3, b_3^1A_1 + b_4^3A_4],$$

$$\pi_1 = [A_1, a_4^0A_0 + a_4^2A_2 + a_4^3A_3, \alpha_2A_0 + \alpha_1A_2], \tag{3}$$

where

$$\alpha_1 = a_4^3a_3^2 + a_{41}^2 - \frac{a_{41}^3a_4^2}{a_4^3}, \quad \alpha_2 = a_4^2a_2^0 + r_{41}^0 - \frac{a_{41}^3a_4^0}{a_4^3}. \tag{4}$$

The axial plane π_2 of the surface (A_2) at the point A_2 is obtained from π_1 by the indicated substitution. The points

$$a_4^2A_2 + a_4^3A_3, \quad b_3^1A_1 + b_4^3A_4 \tag{5}$$

are the points of intersection of the axial plane π_0 , respectively, with the tangent to the line $\omega^1 = 0$ on the surface (A_2) and to the line $\omega^2 = 0$ on the surface (A_1).

The points of intersection of the planes π_0 and π_1 , π_0 and π_2 , are, respectively, the points

$$M_{01} = a_4^0A_0 + a_4^2A_2 + a_4^3A_3, \quad M_{0,-1} = b_3^0A_0 + b_3^1A_1 + b_3^4A_4. \tag{6}$$

These points describe surfaces, which we shall call **surfaces associated with the surface** (A_0) . We shall also say that the surface (M_{01}) is **associated with the congruence** (A_0A_1) , and the point M_{01} is **associated with the line** $[A_0A_1]$. We shall denote the Laplace transformations of the surface (A_0) in the direction $\omega^2 = 0$ by $(A_1), (A_2), \dots$, and in the direction $\omega^1 = 0$ by $(A_{-1}), (A_{-2}), \dots$

Theorem 1. *The point associated with the line $[A_i, A_{i+1}]$ is the intersection of the tangent planes of the surfaces (A_{i-1}) and (A_{i+2}) at the points A_{i-1} and A_{i+2} .*

3. The foci of the axial plane (π_0) form a curve of the second order. The equation of this focal curve with respect to the coordinate triangle determined by the point A_0 and the points (5) is written in the form

$$x_0^2 - a_4^0 x_0 x_1 - b_3^0 x_0 x_2 + (a_4^0 b_3^0 - \alpha_1 \beta_2) x_1 x_2 = 0, \quad (7)$$

where β_2 is obtained from α_1 by the usual substitution.

Theorem 2. *The points (5) and (6) lie on the focal curve (7). The latter means that the axial planes of two neighboring Laplace transformations intersect at a point lying on the focal curves of the axial planes of these Laplace transformations.*

Theorem 3. *The point M_0 is obtained as the intersection of the axial plane π_0 with an infinitely near axial plane under displacement $\omega^1 = 0$ along the surface (A_0) , and as the intersection of the axial plane π_1 with an infinitely near one under displacement $\omega^2 = 0$ along the surface (A_1) .*

4. Surfaces with a focal curve of the axial plane decomposing into a pair of distinct straight lines form two classes, each of which depends on one function of two arguments. For surfaces of the first class ($\alpha_1 = 0$) the focal curve decomposes into the straight lines $[a_4^2 A_2 + a_4^3 A_3, M_{0,-1}]$, $[b_3^1 A_1 + b_3^4 A_4, M_{01}]$. The first straight line is obtained as the intersection of the axial plane π_0 with an infinitely near axial plane under displacement $\omega^2 = 0$ along the surface (A_0) , and under all other displacements along this surface points of the second straight line are obtained. The line $\omega^2 = 0$ on the surface (A_0) is space, on (A_1) the corresponding line is planar, on (A_4) it is a straight line, and the surface (A_4) itself is developable. The Laplace transformation following (A_4) degenerates into a line—the edge of regression of the surface (A_4) . The tangent to the line $\omega^2 = 0$ on the surface $(a_4^2 A_2 + a_4^3 A_3)$ passes through the point A_0 . The axial plane of the surface (A_1) degenerates into the straight line $[A_1 M_{01}]$, carrying two foci: M_{01} and $a_3^2 b_3^0 A_1 - b_1^0 M_{01}$.

The subclass of the first class ($\alpha_1 = \beta_2 = 0$), depending on 10 functions of one argument, is formed by surfaces for which the indicated geometric characteristic holds both in the direction $\omega^2 = 0$ and in the direction $\omega^1 = 0$. The focal curve

of the axial plane π_0 in this case decomposes into the same pair of straight lines, but now the first is obtained as the intersection of the axial plane π_0 with an infinitely near one under displacement $\omega^2 = 0$, and the second—under displacement $\omega^1 = 0$ along the surface (A_0) ; under all

under other displacements along this surface, the axial plane π_0 intersects the infinitely close axial planes at the point $T = a_4^0 b_3^0 A_0 + a_4^0 (b_3^1 A_1 + b_3^4 A_4) + b_3^0 (a_4^2 A_2 + a_4^3 A_3)$. This point T describes a surface (T) , enveloped by the axial planes of the surface (A_0) .

For surfaces of the second class, characterized by the condition $a_4^3 b_3^0 - \alpha_1 \beta_2 = 0$, the focal curve decomposes into the straight lines $[a_4^2 A_2 + a_4^3 A_3, b_3^1 A_1 + b_3^4 A_4]$, $[M_{01}, M_{0,-1}]$. The first straight line is obtained as the intersection of the plane π_0 with the infinitely close axial plane under the displacement $\alpha_1 \omega^1 - a_4^0 \omega^2 = 0$ along the surface (A_0) , the second—under displacements $x_1 \alpha_1 \omega^1 + x_2 b_3^0 \omega^2 = 0$. The first straight line carries one focus, and the second—two.

Surfaces for which the focal curve of the axial plane splits into a pair of coincident straight lines are characterized by the conditions $a_4^0 = b_3^0 = \alpha_1 = 0$ and exist with an arbitrariness of 8 functions of one argument. In this case the points (5) coincide with the points (6). The straight line determined by them serves as the focal curve of the axial plane π_0 and is obtained as the intersection of this plane with the infinitely close one under the displacement $\omega^2 = 0$ along the surface (A_0) . Under any other displacement one obtains the point A_3 , i.e. the infinitely close axial planes of the surface (A_0) form a pencil with center at the point A_3 . The point A_3 lies in the axial plane π_0 , while the surface described by this point degenerates into a line, the tangent to which passes through the point $b_3^1 A_1 + b_3^4 A_4$, serving as the second focus of this tangent. The line (A_3) serves as the edge of regression of the developable surface described by the point $b_3^1 A_1 + b_3^4 A_4$. Surfaces of this kind also possess all the properties of surfaces of the first class ($\alpha_1 = 0$).

A special case of such surfaces ($a_4^0 = b_3^0 = \alpha_1 = \beta_2 = 0$) are surfaces generating a closed five-term Laplace sequence, in which the surfaces (A_3) and (A_4) degenerate into one and the same straight line $A_3 A_4$, serving as the focal curve of the axial plane π_0 and remaining fixed under any displacements along this surface (A_0) . Such surfaces depend on 6 functions of one argument.

5. Surfaces for which the associated surfaces are related by a Laplace transformation will be called **surfaces** L . In order that the surfaces (M_{01}) and $(M_{0,-1})$, associated with the surface (A_0) , be related by a Laplace transformation, it is necessary and sufficient that the equalities

$$a_2^0 b_3^2 + a_4^0 b_3^4 = 0, \quad a_4^1 b_1^0 + a_4^3 j_3^0 = 0. \quad (8)$$

hold.

In this case the conjugate nets of the surfaces (M_{01}) and $(M_{0,-1})$ correspond to the conjugate net of the surface (A_0) .

Theorem 4. *Surfaces L exist with an arbitrariness of 12 functions of one argument.*

Theorem 5. *All focal surfaces of the Laplace sequence generated by a surface L are surfaces L .*

The Laplace sequence generated by a surface L will be called a **sequence L** . By virtue of Theorem 5, the surfaces associated with the surfaces of the sequence L themselves form a Laplace sequence, which we shall call the **associated sequence of the original sequence**.

Theorem 6. *The Laplace sequence associated with a sequence L is itself a sequence L .*

The proof of Theorems 5 and 6 is based on verifying conditions analogous to equalities (8) for the surfaces (A_1) and (M_{01}) .

6. Let us note three special cases of surfaces L .

A. $a_4^0 = b_3^2 = 0$, $a_4^1 b_1^0 + a_4^3 b_3^0 = 0$. The arbitrariness of existence is equal to 11 functions of one argument. For such surfaces, the tangents to the line $\omega^2 = 0$ on the surface (A_2) and to the line $\omega^1 = 0$ on the surface (A_{-2}) intersect, respectively, the planes $[A_1 A_{-1} A_{-2}]$ and $[A_0 A_1 A_2]$; the line $[A_0 M_{01}]$ is tangent to the focal curve of the axial plane π_0 , and both of the indicated properties are preserved for all surfaces of the Laplace sequence.

B. $a_4^0 = b_3^0 = a_4^1 = b_3^2 = 0$. The arbitrariness of existence is equal to 10 functions of one argument. The tangent to the line $\omega^2 = 0$ on the surface (A_2) intersects the line $A_1 A_{-2}$, while the tangent to the line $\omega^1 = 0$ on the surface (A_{-2}) intersects the line $A_1 A_2$. The lines $A_0 M_{01}$ and $A_0 M_{0,-1}$ are tangent to the focal curve of the axial plane π_0 . These properties hold for any surface of the Laplace sequence. The first shows that the original surface is a generalized Slotnik surface ⁽¹⁾.

Both in case A and in case B, the associated Laplace sequence is of the same type as the original one and is inscribed in it.

C. $a_4^0 = b_3^0 = a_4^1 = b_3^2 = a_4^2 = b_3^1 = 0$. We have the most general closed five-membered Laplace sequence, depending on 8 functions of one argument. Its associated sequence coincides with it itself.

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CITED LITERATURE

¹ S. P. Finikov, *The Method of Exterior Forms*, Moscow-Leningrad, 1948.

Note: Figure translations are in progress. See original paper for figures.

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