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Abstract

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On Extremal Problems for Functions Satisfying Additional Constraints Inside a Domain, and on the Application of These Problems to Questions of Approximation

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In this article an attempt is made to give a general approach to the study (primarily from the qualitative side) of extremal problems for classes of functions satisfying a certain restriction on growth in the domain of their definition and an additional restriction on growth on some subset of the domain of definition. A typical representative of the problems considered in this paper is the following. In a domain G a class of functions $f(z)$, analytic in G, is given, satisfying there the condition

$$|f(z)| \leqslant 1, \qquad z \in G, \tag{0,1}$$

and the additional condition

$$|f(z)| \leqslant \varepsilon < 1, \qquad z \in D, \tag{0,2}$$

where D is some subset of G. It is required to find

$$\sup |f(z_0)|, \qquad \sup |f'(z_0)| \quad \text{etc.} \tag{0,3}$$

over the functions of this class and to investigate the properties of the extremal function. Problems of this type were first considered by Milloux ([1]). Further results and literature on various questions adjacent to this formulation may be found in ([2–5]). Important results obtained recently by S. N. Mergelyan, A. L. Shaginian, M. M. Dzhrbashyan, M. M. Lavrentiev, concerning various problems (both for analytic and for harmonic functions) connected with the general formulation stated at the beginning of the article, and the use of these problems in other problems of the theory of functions and the theory of differential equations, are presented in the surveys ([6,7]). We also note the papers ([8–12]).

§1. It is clear that the majority of problems of type (0,1)–(0,3) fit into the scheme contained in the following two theorems. (These theorems are dual to each other; sometimes one of them gives more complete results, sometimes the other.)

Theorem 1. Let E and E_1 be linear locally convex topological spaces; $\mathcal{E} \subset E$ a subspace in E; $N\varphi$ a continuous linear operator from E_1 into E; $f_0 \in E$ an arbitrary element; $P(f)$ and $P_1(\varphi)$ continuous symmetric convex functionals in E and E_1, respectively. Denote by T the set of linear functionals $l(f)$ on E satisfying the conditions

$$l(f) = 0, \quad f \in \mathcal{E}; \qquad |l(F)| \leqslant P(F), \quad F \in E; \qquad |l(N\varphi)| \leqslant P_1(\varphi), \quad \varphi \in E_1. \tag{1,1}$$

Then

$$\sup_{l \in T} |l(f_0)| = \inf_{\substack{f \in \mathcal{E} \\ \varphi \in E_1}} [P(f_0 - f - N\varphi) + P_1(\varphi)]. \tag{1,2}$$

There always exists an extremal functional $l^* \in T$.

Theorem 2. Let E and E_1 be linear locally convex topological spaces; $\mathcal{E}$ a subspace in E; A a symmetric convex body in E, containing the origin as an interior point; $B = A \cap \mathcal{E}$; Nf a linear continuous operator from E into the space E_1, endowed with the weak topology; S a symmetric convex body in E_1, containing the origin as an interior point. Denote by B^s the set of elements f for which

$$f \in B; \qquad Nf \in S, \tag{1,3}$$

and by $\mathfrak{M}$ the set of linear functionals from E^* that are equal to zero on $\mathcal{E}$. For arbitrary $l_0 \in E^*$ we have

$$\sup_{f \in B^s} |l_0(f)| = \inf_{\substack{\Lambda \in E_1^* \\ m \in \mathfrak{M}}} \left[\sup_{f \in A} |(l_0 - \tilde{l} - m)(f)| + \sup_{\varphi \in S} |\Lambda(\varphi)| \right], \tag{1,4}$$

the infimum in (1,4) is always attained (E^* and E_1^* are the spaces conjugate to E and E_1; $\tilde{l}(f) = \Lambda(Nf)$).

For the proof of Theorem 1 one should consider the product $\widetilde{E} = E \times E_1$ and in it the convex functional

$$\mathcal{P}(F) = \mathcal{P}(f; \varphi) = P(f) + P_1(\varphi); \qquad f \in E, \quad \varphi \in E_1.$$

In the space $\widetilde{E}^*$ conjugate to $\widetilde{E}$, consider the subspace K^*, consisting of functionals $L(F)$ of the form $L = (l, \widetilde{\Lambda})$, where $l \in E^*$ is arbitrary and $\widetilde{\Lambda}\varphi = l(N\varphi)$. If $K \subset \widetilde{E}$ is the set of all zeros of the functionals from K^*, then, by means of the relation

$$\inf_{F \in K} \mathcal{P}(F_0 - F) = \sup_{\substack{L \in K^* \\ |L(F)| \in \mathcal{P}(F)}} |L(F_0)|, \qquad (1,5)$$

putting $F_0 = (f_0, 0)$, one can obtain (1,2).

Further, we investigated additional questions of existence of extremal elements in (1,2) and (1,4), characteristic criteria for extremal elements, and questions of uniqueness. These results, relating to Theorem 1 (in its more particular case), are set forth in ([13]).

§ 2. Let G be a finitely connected domain with rectifiable boundary Γ; $\rho(\zeta)$ a weight function satisfying the inequalities

$$0 < m \le \rho(\zeta) \le M < +\infty, \qquad \zeta \in \Gamma; \qquad (2,1)$$

$B_{\rho(\zeta)}$ is the class of single-valued bounded analytic functions in G, for which

$$|f(\zeta)| \le \rho(\zeta) \qquad (2,2)$$

almost everywhere on Γ.

Theorem 3. Let $B_{\rho(\zeta),\varepsilon}$ be the subclass of $B_{\rho(\zeta)}$ consisting of functions satisfying, in addition to (2,2), the inequality (0,2), where $\overline{D} \subset G$. For any summable $\omega(\zeta)$ on Γ

$$\sup_{f \in B_{\rho,\varepsilon}} \left| \int_\Gamma f(\zeta)\omega(\zeta)\, d\zeta \right| = \inf_{\substack{\lambda \\ \varphi \in E_1(G)}} \left[\int_\Gamma \rho(\zeta) \left| \omega(\zeta) - \int_D \frac{d\lambda}{t - \zeta} - \varphi(\zeta) \right| ds + \varepsilon \int_D |d\lambda| \right], \qquad (2,3)$$

where $d\lambda$ is an arbitrary measure on D, and the class $E_1(G)$ consists of functions representable by the Cauchy integral through their boundary values. Relations of type (2,3) also occur when (0,2) is replaced by other conditions, for example of integral character. In certain cases one may allow $\overline{D} \cap \Gamma \ne 0$; one may also, instead of the class $B_{\rho(\zeta)}$, proceed from the classes E_p, etc.

On the basis of (2,3), extremal functions in (2,3) are investigated. The results are analogous to those obtained in ([14,15]) (see also the survey ([16])).

Theorem 4. In order that $f^*(z)$, $\varphi^*(z)$, λ^* be extremal in (2,3) (with $\overline{D} \subset G$), it is necessary and sufficient that the relations

$$f^*(\zeta)\left[\omega(\zeta)-\int_D\frac{d\lambda^*}{t-\zeta}-\varphi^*(\zeta)\right]d\xi=e^{i\alpha}\rho(\zeta)\left|\omega(\zeta)-\int_D\frac{d\lambda^*}{t-\zeta}-\varphi^*(\zeta)\right|ds \tag{2,4}$$

hold almost everywhere on Γ, and

$$f^*(t)\,d\lambda^*=e^{i\alpha}\xi\,|d\lambda^*| \tag{2,5}$$

almost everywhere with respect to the measure $|d\lambda^*|$ on D; $f^*(z)$ is unique up to the factor $e^{i\alpha}$.

Theorem 5. If $\omega(\zeta)$ is the boundary value on Γ of a function $\omega(z)$ analytic in some annular domain $\widetilde{G}\subset G$ adjacent to Γ, then

$$f^*(z)=\exp\left[\frac{1}{2\pi}\int_\Gamma\ln\rho(\zeta)\frac{\partial g(\zeta,z)}{\partial n}\,ds+iV(z)-\sum_{k=1}^m p(z,z_k)\right],$$

where $g(\zeta,z)$ is the Green' s function for G; $p(z,z_k)$ is the complex Green' s function for G; $V(z)$ is the harmonic conjugate to the integral preceding it.

Theorem 6. If, under the conditions of Theorem 5, $\rho(\zeta)\equiv 1$, then $f^*(z)$ maps G onto the unit disk covered $m\geq n$ times (n is the connectivity of G).

Theorems 5 and 6 hold under considerably more general constraints inside the domain than (0,2).

Theorem 7. Suppose that, under the conditions of Theorem 6, G is the unit disk and D consists of a finite number k of arcs of circles, with ε in inequality (0,2) being its own on each of the arcs, and suppose that $r_j^m\neq\varepsilon_j$ for no integer $m\geq 0$, where r_j is the radius of the arc numbered j, and ε_j is the number assigned on this arc in (0,2).

Then there exists a finite subset $D_N\subset D$, consisting of points $a_1,\dots,a_N$, such that the extremal problem with constraints (0,2) on D_N has the same solution as the extremal problem with constraints (0,2) on all of D. Moreover, the number m of coverings from Theorem 6 and the numbers k,N are connected by the relation

$$N\leq km. \tag{2,6}$$

If, however,

$$\omega(\zeta)=\sum_{\mu=1}^{s}\sum_{l=1}^{l_\mu}\frac{C_{\mu l}}{(\zeta-b_\mu)^l},$$

then

$$N \geq m + 1 - \sum_{\mu=1}^{s} l_\mu. \tag{2,7}$$

In particular, for $k = 1$, for the problem on $\sup|f(z_0)|$, we obtain

$$N = m. \tag{2,8}$$

This special case was studied in detail by Hayman ([4]).

Theorem 7 is an analogue of a well-known fact in the theory of Chebyshev approximations.

§ 3. Relations of the type (2,3) can also be applied to the establishment of certain quantitative relations.

Theorem 8. *If* $|f(z)| \leqslant 1$ *for* $|z| < 1;\ |f(\alpha_j)| \leqslant \varepsilon_j < 1,\ |\alpha_j| < 1,\ j = 1, \dots, n,$ *then*

$$|f(z)| \leqslant \prod_1^n \left|\frac{z-\alpha_j}{1-\overline{\alpha}_j z}\right| \left[1 + \sum_1^n \varepsilon_j \left(1 - \left|\frac{z-\alpha_j}{1-\overline{\alpha}_j z}\right|^2\right) \left|\frac{1-\overline{\alpha}_j z}{z-\alpha_j}\right| \prod_{\substack{k=1\\k\neq j}}^n \left|\frac{1-\alpha_k\overline{\alpha}_j}{\alpha_j-\alpha_k}\right|\right]. \tag{3,1}$$

If $\varepsilon_1 = \dots = \varepsilon_n = \varepsilon$, then a qualitatively worse estimate was found earlier by Whittaker [(17)]. In Whittaker' s inequality the majorant tends to infinity when $|z| \to 1$. For the proof of (3,1) one uses the solutions of extremal problems from [(18)], pp. 260–261. Analogous inequalities also hold in the classes $H_p,\ p \geqslant 1$.

§ 4. With the aid of the estimates following from inequality (3,1), one can prove the following uniqueness theorem.

Theorem 9. *Let* $|a_n| < 1,\ n = 1, 2, \dots;$

$$\sum_1^\infty (1 - |a_n|) = \infty, \qquad \frac{|a_n| - |a_{n-1}|}{(1-|a_n|)(1-|a_{n-1}|)} \geqslant d > 0, \tag{4,1}$$

where $d > 0$ *does not depend on* n. *If the bounded function* $f(z)$ *satisfies the condition*

$$\lim_{n\to\infty} \ln\{|f(a_n)|(1-|a_n|)\} = -\infty, \tag{4,2}$$

then $f(z) \equiv 0$.

This theorem somewhat strengthens the theorem of I. V. Ushakova [12] and Theorem 2 from [8] of A. A. Shaginyan.

Theorem 9 immediately extends to meromorphic functions of bounded type. The sharpness of Theorem 9 follows from the fact that the bounded function

$$e^{-\frac{1}{1-z}} \neq 0.$$

Uniqueness theorems of the type of Theorem 9 are connected with such approximation theorems in which it is possible to take account of the magnitudes of the coefficients of the approximating aggregates. This connection follows from the general Theorems 1 and 2. For lack of space we do not present these results.

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Note: Figure translations are in progress. See original paper for figures.