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B. S. RIMSKII-KORSAKOV

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Abstract

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B. S. RIMSKII-KORSAKOV

ON AN ANALOGUE OF THE POISSON FORMULA FOR COSINE FOURIER TRANSFORMS

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1°. In the theory of Fourier transforms the Poisson formula is known (^{1}):

$$\sqrt{x}\left\{\frac{1}{2}f(0)+\sum_{n=1}^{\infty}f(nx)\right\}=\sqrt{\frac{2\pi}{x}}\left\{\frac{1}{2}F_c(0)+\sum_{n=1}^{\infty}F_c\left(\frac{2\pi n}{x}\right)\right\},$$

where

$$F_c(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f(t)\cos xt\,dt.$$

In his work (^{2}) Meijer obtained an analogue of the Poisson formula for an integral transform, first considered by Bhatnagar (^{3}), with Fourier kernel

$$\tilde{\omega}_{\mu,\nu}(x)=\sqrt{x}\int_0^{\infty}J_\mu(t)J_\nu\left(\frac{x}{t}\right)\frac{dt}{t}\qquad\left(\mu,\nu>-\frac{1}{2}\right),$$

where $J_\nu(t)$ is the Bessel function of the first kind.

The present article is devoted to the construction of another analogue of the Poisson formula, connected with generalized integral Fourier transforms, for sums of the form

$$\sum_{n=1}^{\infty}F(n)f(nx),$$

where $F(n)$ is the number of integral ideals with norm n of an algebraic field Ω, with fundamental number Δ, of order $\chi=r_1+2r_2$ (r_1 is the number of real fields conjugate to Ω, r_2 is half the number of imaginary conjugate fields). Below we consider the rational field ($r_1=1,\ r_2=0$), an imaginary quadratic field ($r_1=0,\ r_2=1$), and a real quadratic field ($r_1=2,\ r_2=0$).

2°. To each of the fields under consideration there corresponds a function

$$\frac{2}{\pi}L(x) = \frac{2}{\pi}L_{r_1,r_2}(x),$$

defined by the equality

$$G(1-s) = \frac{\Gamma^{r_1}(s/2)\Gamma^{r_2}(s)}{\Gamma^{r_1}((1-s),2)\Gamma^{r_2}(1-s)} = \int_0^\infty \frac{2}{\pi}L(x)x^{s-1}\,dx, \qquad 0 < \operatorname{Re} s < \frac{3}{4}.$$

The functions $\frac{2}{\pi}L_{r_1,r_2}(x)$ are Fourier kernels. This is easy to see, taking into account that

$$\frac{2}{\pi}L_{1,0}(x) = \frac{2}{\sqrt{\pi}}\cos 2x, \qquad \frac{2}{\pi}L_{0,1}(x) = \tilde{\omega}_{1/2,-1/2}(x) = J_0(2\sqrt{x}),$$

$$\frac{2}{\pi}L_{2,0}(x) = 2\tilde{\omega}_{-1/2,-1/2}(4x) = \frac{4}{\pi}\left\{K_0(4\sqrt{x}) - \frac{\pi}{2}Y_0(4\sqrt{x})\right\},$$

where $K_0(x)$ is the Macdonald function, and $Y_0(x)$ is the Neumann function.

3°. Theorem. If: 1) the function $\mathfrak{F}(s)$ of the complex variable $s = \sigma + it$ is regular in the strip $-\alpha \le \sigma \le 1 + \beta$ $(0 < \alpha < 1;\ \beta > 0)$, except for the point $s = 0$, where it has a pole of first order; 2) symmetri-

there is a $c > 0$ such that, uniformly with respect to σ in the indicated strip, $\mathfrak{F}(s) = O(e^{-c|t|})$ as $|t| \to \infty$; 3) there exists the finite limit $\lim\limits_{s\to 0}\dfrac{\mathfrak{F}(s)}{\Gamma(s)}$; 4)

$$f(x) = \frac{1}{2\pi i}\int_{1+\beta-i\infty}^{1+\beta+i\infty} \mathfrak{F}(s)x^{-s}\,ds, \quad x > 0,$$

then

$$\sum_{n=1}^{\infty} F(n)f(nx) = R_0 + R_1 + \frac{1}{Ax}\sum_{n=1}^{\infty} F(n)\int_0^\infty f(t)\frac{2}{\pi}L\left(\frac{nt}{A^2x}\right)dt, \tag{1}$$

where

$$R_0 = \zeta_\Omega(0)\lim_{s\to 0}\{s\mathfrak{F}(s)\}, \qquad R_1 = \frac{\mathfrak{F}(1)}{x}\lim_{s\to 1}\{(s-1)\zeta_\Omega(s)\},$$

$\zeta_\Omega(s)$ is the Dedekind zeta-function, $A = \sqrt{|\Delta|}/2^r\pi^{\chi/2}$.

Proof. Consider the series

$$\sum_{n=1}^{\infty} F(n)f(nx) = \sum_{n=1}^{\infty} F(n)\frac{1}{2\pi i}\int_{1+\beta-i\infty}^{1+\beta+i\infty} \mathfrak{F}(s)(nx)^{-s}\,ds. \tag{2}$$

On the right-hand side of (2) it is permissible to change the order of integration and summation, by virtue of the finiteness of

$$\int_{-\infty}^{\infty} |\mathfrak{F}(1+\beta+it)|\, dt,$$

which follows from the estimate in the second condition of the theorem, and of the absolute convergence, for $\beta > 0$, of the series

$$\sum_{n=1}^{\infty} \frac{F(n)}{n^{1+\beta}}.$$

Therefore

$$\sum_{n=1}^{\infty} F(n) f(nx) = \frac{1}{2\pi i} \int_{1+\beta-i\infty}^{1+\beta+i\infty} \mathfrak{F}(s) \left\{ \sum_{n=1}^{\infty} \frac{F(n)}{n^s} \right\} x^{-s}\, ds =$$

$$= \frac{1}{2\pi i} \int_{1+\beta-i\infty}^{1+\beta+i\infty} \mathfrak{F}(s) \zeta_\Omega(s) x^{-s}\, ds. \tag{3}$$

Since the Dedekind zeta-function is regular in the whole plane except at the point $s = 1$, where it has a pole of the first order, the function

$$\varphi(s) = \mathfrak{F}(s) \zeta_\Omega(s) x^{-s},$$

by the first condition of the theorem, is regular in the strip $-\alpha < \sigma < 1 + \beta$ $(0 < \alpha < 1,\ \beta > 0)$, except at the points $s = 0$ and $s = 1$, where it has first-order poles with residues

$$R_0 = \zeta_\Omega(0) \lim_{s\to 0} \{s \mathfrak{F}(s)\}, \qquad R_1 = \frac{\mathfrak{F}(1)}{x} \lim_{s\to 1} \{(s-1)\zeta_\Omega(s)\}.$$

Therefore, if Γ is the perimeter of the rectangle with vertices at the points $A(1+\beta-it)$, $B(1+\beta+it)$, $C(-\alpha+it)$, $D(-\alpha-it)$, then

$$\frac{1}{2\pi i} \oint_\Gamma \varphi(s)\, ds = R_0 + R_1.$$

Hence it follows that

$$\frac{1}{2\pi i} \int_{AB} \varphi(s)\, ds = R_0 + R_1 + \frac{1}{2\pi i} \int_{DC} \varphi(s)\, ds - \frac{1}{2\pi i} \left(\int_{BC} \varphi(s)\, ds + \int_{DA} \varphi(s)\, ds \right).$$

In the strip under consideration

$$x^{-s} = O(1) \quad \text{for fixed } x > 0;$$

$$\zeta_\Omega(s) = O(t^Q) \quad \text{uniformly, where } Q \text{ depends on } -\alpha\ (^4);$$

therefore, also taking into account the second condition of the theorem, we have

$$\varphi(s) = O\left(e^{-c|t|}t^{Q}\right) \quad \text{as } |t| \to \infty.$$

Since t does not change on BC,

$$\int_{BC} \varphi(s)\,ds = \int_{BC} O\left(e^{-c|t|}t^{Q}\right)\,ds = O\left(e^{-c|t|}t^{Q}\right) \to 0 \quad \text{as } |t| \to \infty;$$

similarly,

$$\int_{DA} \varphi(s)\,ds \to 0 \quad \text{as } |t| \to \infty.$$

Therefore, after passing to the limit as $|t| \to \infty$, equality (3) may be rewritten as

$$\sum_{n=1}^{\infty} F(n)f(nx) = R_0 + R_1 + \frac{1}{2\pi i}\int_{-a-i\infty}^{-a+i\infty} \mathcal{F}(s)\zeta_\Omega(s)x^{-s}\,ds,$$

or, after the obvious change of variable, also as

$$\sum_{n=1}^{\infty} F(n)f(nx) = R_0 + R_1 + \frac{1}{2\pi i}\int_{1+\alpha-i\infty}^{1+\alpha+i\infty} \mathcal{F}(1-s)\zeta_\Omega(1-s)x^{s-1}\,ds.$$

Taking into account the functional equation of the Dedekind zeta-function

$$\zeta_\Omega(1-s) = A^{2s-1}G(1-s)\zeta_\Omega(s),$$

we obtain

$$\sum_{n=1}^{\infty} F(n)f(nx) = R_0 + R_1 + \frac{1}{2\pi i}\int_{1+\alpha-i\infty}^{1+\alpha+i\infty} \mathcal{F}(1-s)A^{2s-1}G(1-s)\zeta_\Omega(s)x^{s-1}\,ds =$$

$$= R_0 + R_1 + \frac{1}{Ax}\frac{1}{2\pi i}\int_{1+\alpha-i\infty}^{1+\alpha+i\infty} \mathcal{F}(1-s)G(1-s)\sum_{n=1}^{\infty}\frac{F(n)}{n^s}\left(\frac{1}{A^2x}\right)^{-s} ds. \quad (4)$$

On the right-hand side of (4) it is permissible to change the order of summation and integration, by virtue of the absolute convergence of the series

$$\sum_{n=1}^{\infty}\left|\frac{F(n)}{n^{1+\alpha}}\right|$$

and the finiteness of the integral

$$\int_{-\infty}^{\infty}\left|\mathcal{F}(-\alpha-it)G(-\alpha-it)\left(\frac{1}{A^2x}\right)^{-(1+\alpha+it)}\right|dt.$$

This last fact follows from the second condition of the theorem and from the validity, on the line $s=1+\alpha+it$ for $|t|\geq 1$, of the estimate

$$G(1-s)=O\left(|t|^{\chi(\alpha+1/2)}\right),$$

obtained, on the basis of the definition of the function $G(s)$, with the help of the known properties of the gamma-function and the estimates

$$\left|\sin\frac{\pi s}{2}\right|=O\left(e^{\frac{1}{2}\pi|t|}\right),\qquad \cos\frac{\pi s}{2}=O\left(e^{\frac{1}{2}\pi|t|}\right),$$

$$\Gamma(s)=O\left(e^{-\frac{1}{2}\pi|t|}|t|^{\sigma-1/2}\right),\qquad -1\leq\sigma\leq 2,\ |t|\geq 1. \tag{5}$$

Changing in (4) the order of summation and integration, we obtain:

$$\sum_{n=1}^{\infty}F(n)f(nx)=R_0+R_1+\frac{1}{Ax}\sum_{n=1}^{\infty}F(n)\frac{1}{2\pi i}\int_{1+\alpha-i\infty}^{1+\alpha+i\infty}\mathcal{F}(1-s)G(1-s)\left(\frac{n}{A^2x}\right)^{-s}ds. \tag{5}$$

By virtue of the third condition of the theorem, the value of the integral on the right-hand side of (5) will not change if, as the contour of integration, we take the straight line $s=\gamma+it$ $(0<\gamma<{}^3/_4,\ -\infty<t<\infty)$. Therefore

$$\sum_{n=1}^{\infty}F(n)f(nx)=R_0+R_1+\frac{1}{Ax}\sum_{n=1}^{\infty}F(n)\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\mathcal{F}(1-s)G(1-s)\left(\frac{n}{A^2x}\right)^{-s}ds. \tag{6}$$

In the theory of Mellin transforms the formula

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}A(1-s)B(s)c^{-s}ds=\int_0^{\infty}a(x)b(cx)\,dx$$

is known (where A and B are the Mellin transforms of the functions a and b, respectively), valid when certain known conditions are satisfied (^{6}). Since in the case under consideration these conditions are satisfied, equality (6) can be rewritten in the form (1), as was required to prove.

Remark. A consequence of (1) is the expansion of Koshlyakov's sigma-function into partial fractions (^{7}):

$$\sigma(x) = \frac{1}{A\pi}\sum_{n=1}^{\infty} F(n)K_{r_1,r_2}\left(\frac{nx}{A^2}\right) = R_0 + R_1 + \frac{x}{\pi}\sum_{n=1}^{\infty}\frac{F(n)}{x^2+n^2}, \qquad x>0. \quad (7)$$

To verify this, one should note (taking into account formulas 2.4–6 from (^{7}) and 8.13(2), 10.3(50) from (^{8})) that, for

$$f(x) = \frac{1}{A\pi}K_{r_1,r_2}\left(\frac{x}{A^2}\right) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{A^{2s-1}}{2\cos\frac{\pi s}{2}}G(1-s)x^{-s}ds, \qquad \alpha>0, \quad x>0,$$

$$\frac{1}{Ax}\int_0^{\infty} f(t)\frac{2}{\pi}L\left(\frac{nt}{A^2x}\right)dt = \frac{1}{\pi}\frac{x}{x^2+n^2}, \quad (8)$$

independently of which field is being considered (rational, imaginary quadratic, or real quadratic).

Another consequence of (1) is also the following formula of N. S. Koshlyakov (^{7}), p. 119):

$$\sum_{n=1}^{\infty} F(n)X_{r_1,r_2}\left(\frac{nx}{A}\right) = R_0 + R_1 + \frac{1}{x}\sum_{n=1}^{\infty} F(n)X_{r_1,r_2}\left(\frac{n}{Ax}\right), \qquad x>0, \quad (9)$$

where

$$X(x) = X_{r_1,r_2}(x) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\Gamma^{r_1}\left(\frac{s}{2}\right)\Gamma^{r_2}(s)x^{-s}ds, \qquad \alpha>0, \quad x>0.$$

To verify this, one should note (using formulas (4.3) from (^{7}) and 1.4(11), 8.6(10), 9.4(25), 10.3(52) from (^{8})) that

$$\int_0^{\infty} X(t)\frac{2}{\pi}L(xt)\,dt = X(x),$$

i.e. the functions $X_{r_1,r_2}(x)$ are self-reciprocal under integral transformations with Fourier kernels $\frac{2}{\pi}L_{r_1,r_2}(x)$.

All-Union Correspondence
Power Engineering Institute

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Note: Figure translations are in progress. See original paper for figures.

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