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Abstract

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ON THE THEORY OF CAUCHY-POISSON WAVES NEAR AN INCLINED SHORE

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A plane problem is considered concerning the unsteady motions of a heavy fluid near a shore with angle of inclination $\pi/2n$, where n is an integer. Asymptotic expressions are obtained for the elevation of the free surface in the case when the initial impulse or the initial elevation is concentrated near the point of intersection of the free surface with the shore.

1. It is known $(^1)$ that in an incompressible fluid occupying the space between the free surface and a shore with angle of inclination $\pi/2n$, where n is an integer, there exists a system of standing waves

$$\varphi_1(mx, my; \sqrt{mt}) = \cos\sqrt{mgt}\,\frac{\pi}{(n-1)!\sqrt{n}}\;\mathrm{Re}\sum_{k=1}^{n} c\,e^{mz\beta_k}, \qquad (1{,}1)$$

where

$$\beta_k = \exp\left\{i\pi\left(\frac{k}{n}+\frac{1}{2}\right)\right\},$$

$$c_k = \exp\left\{i\pi\left(\frac{n+1}{4}-\frac{k}{2}\right)\right\}\mathrm{ctg}\,\frac{\pi}{2n}\,\mathrm{ctg}\,\frac{2\pi}{2n}\cdots\mathrm{ctg}\,\frac{(k-1)\pi}{2n}\qquad (k=2,3,\dots,n),$$

$$c_1 = c_n, \qquad z = x+iy.$$

Multiplying (1,1) by an arbitrary function of the parameter $F_1(m)$ and integrating with respect to m from 0 to ∞, we obtain the more general solution

$$\varphi(x,y,t) = \int_0^\infty F_1(m)\,\varphi_1(mx, my; \sqrt{mt})\,dm. \qquad (1{,}2)$$

The function φ satisfies the condition of constant pressure on the free surface and the condition of no flow through the inclined bottom. We choose the function $F_1(m)$ so that the initial condition is satisfied

$$\varphi(x,0;0) = \frac{1}{\rho}F(x), \tag{1,3}$$

where $F(x)$ is the distribution of impulse over the free surface at $t = 0$. Substituting (1,2) into (1,3), we obtain a Fredholm integral equation of the first kind for determining the function $F_1(m)$,

$$\frac{1}{\rho}F(x) = \int_0^\infty F_1(m)\,\varphi_1(mx,0;0)\,dm. \tag{1,4}$$

The integral equation (1,4) can be solved using Mellin transforms (see (^{2}), p. 401). Multiplying both sides of (1,4) by x^{s-1} and integrating with respect to x over the interval from 0 to ∞, we obtain

$$\frac{1}{\rho}\,\bar{F}(s) = \bar{F}_1(1-s)\,\bar{\varphi}_1(s), \tag{1,5}$$

where the barred functions are the Mellin transforms of the corresponding unbarred functions, constructed by the formula

$$\bar{f}(s) = \int_0^\infty f(x)x^{s-1}\,dx.$$

From (1.5) we obtain

$$\bar{F}_1(s) = \frac{1}{\rho}\frac{\bar{F}(1-s)}{\varphi_1(1-s)}. \tag{1.6}$$

Applying to (1.6) the inversion formula for the Mellin transform and substituting the result in (1.2), we obtain the solution of the desired problem

$$\varphi(x,y;t) = \frac{1}{2\pi i\rho}\int_0^\infty \varphi_1(mx,\ my;\ \sqrt{m}\,t)\int_0^\infty F(x)\int_{c-i\infty}^{c+i\infty}\frac{(mx)^{-s}\,ds}{\varphi_1(1-s)}\,dx\,dm. \tag{1.7}$$

2. For $n = 1$, i.e., when the shore is perpendicular to the free surface, by symmetry the problem formulated above is the classical plane Cauchy-Poisson wave problem on the surface of a fluid of infinite depth. In this case (1.4) is the ordinary Fourier cosine transform. In [3] it is shown that for $n = 2$, (1.4) is a generalized Fourier transform and, consequently, the inversion formula in this case is also symmetric. In the general case, for integer $n > 2$, one must use formula (1.7). The inner integral in (1.7), for small x, can be computed using Cauchy' s theorem. The contour of

> integration, passing parallel to the imaginary axis, may be closed by a part of a circle with center at the origin, situated to the left of the contour. It is easy to see that, according to Jordan' s lemma, the integral over the circle tends to zero as the radius increases to infinity, and that the desired integral is equal to the sum of the residues at the poles of the integrand. The integral computed in this way will be a series in powers of mx, which must then be integrated with respect to x and m over the limits from 0 to ∞.

Consider the special case in which the initial impulse is concentrated near the origin, i.e., $F(x) = \delta(x)$, where $\delta(x)$ is the Dirac δ-function. By the properties of the δ-function, the integral with respect to x in (1.7) is equal to the term of the series with the zeroth power of mx. From physical considerations it is clear that this term must be different from zero. Thus, for this special case, the solution of the problem reduces to finding the residue of the integrand $(mx)^{-s}/\varphi_1(1-s)$ at the point $s = 0$. According to what was said above, for $n = 1, 2$ the solution is known. We therefore give the results of the computations for the cases $n = 3, 4$ (analogous calculations can be carried out for any n):

$$\varphi(x, y; t) = -\frac{1}{\pi\rho\sqrt{3}}\int_0^\infty \cos\sqrt{mg}\,t\,\times$$

$$\times\operatorname{Re}\Big[ie^{mz\exp i\pi5/6} + \sqrt{3}\,e^{mz\exp i\pi7/6} - ie^{mz\exp i\pi3/2}\Big]\,dm,\qquad n = 3; \tag{2.1}$$

$$\varphi(x, y; t) = -\frac{4}{\pi\rho}\int_0^\infty \cos\sqrt{mg}\,t\;\operatorname{Re}\Big[(-1+i)e^{mz\exp i\pi3/4} +$$

$$+(\sqrt{2}+1)(1+i)e^{mz\exp i\pi} + (\sqrt{2}+1)(1-i)e^{mz\exp i\pi5/4} - (1+i)e^{mz\exp i\pi3/2}\Big]\,dm,\qquad n = 4. \tag{2.2}$$

It is easy to construct the corresponding expressions for the potential $\varphi'(x, y; t)$ of the flow under the action of an initial elevation of the surface, specified in the form of a δ-function. φ' is related to φ by the relation $\partial\varphi'/\partial t = \rho g\varphi$. Setting $y = 0$ in the formulas obtained and expanding the integrands in series, we find expressions for the elevation of the free surface $\eta(x, t)$:

$$\eta(x, t) = \frac{1}{g}\frac{\partial\varphi'(x, 0; t)}{\partial t} = -\frac{1}{\pi x\sqrt{3}}\left[\sin\frac{\pi}{6} - \omega\frac{1!}{2!}\sin 2\frac{\pi}{6} + \omega^2\frac{2!}{4!}\sin 3\frac{\pi}{6} - \cdots\right.$$

$$\left.\cdots + \sqrt{3}\left(\cos\frac{\pi}{6} - \omega\frac{1!}{2!}\cos 2\frac{\pi}{6} + \omega^2\frac{2!}{4!}\cos 3\frac{\pi}{6} - \cdots\right) - \left(1 - \omega^2\frac{2!}{4!} + \omega^4\frac{4!}{8!} - \cdots\right)\right],\qquad n = 3; \tag{2,3}$$

$$\eta(x,t) = -\frac{4}{\pi x}\left[\sqrt{2}\left(\cos\frac{\pi}{4} - \omega\frac{1!}{2!}\cos 2\frac{\pi}{4} + \omega^2\frac{2!}{4!}\cos 3\frac{\pi}{4} - \cdots\right) - \right.$$

$$-(\sqrt{2}+2)\left(\sin\frac{\pi}{4} - \omega\frac{1!}{2!}\sin 2\frac{\pi}{4} + \omega^2\frac{2!}{4!}\sin 3\frac{\pi}{4} - \cdots\right) +$$

$$+(\sqrt{2}+1)\left(1 - \omega\frac{1!}{2!} + \omega^2\frac{2!}{4!} - \cdots\right) -$$

$$\left. -\left(1 + \omega\frac{1!}{2!} - \omega^2\frac{2!}{4!} - \omega^3\frac{3!}{6!} + \omega^4\frac{4!}{8!} + \omega^5\frac{5!}{10!} - \cdots\right)\right], \qquad n = 4, \qquad (2,4)$$

where $\omega = gt^2/x$. To obtain the corresponding expressions for the case of an initial impulse, it is only necessary to apply the operator

$$\frac{1}{\rho g}\frac{\partial}{\partial t}.$$

The series obtained converge rapidly for small values of ω. Asymptotic expressions valid for large values of ω can be obtained. Applying the method of stationary phase, we have

$$\eta(x,t) = \frac{1}{g}\frac{\partial\varphi'}{\partial t} = \frac{2}{x\sqrt{3}}\sqrt{\frac{\omega}{2\pi}}\left\{e^{-\omega\sqrt{3}/2}\left[\sin\left(\frac{\omega}{2} - \frac{\pi}{4}\right) - \sqrt{3}\cos\left(\frac{\omega}{2} - \frac{\pi}{4}\right)\right] - \frac{1}{\sqrt{8}}\sin\left(\frac{\omega}{4} - \frac{\pi}{4}\right)\right\}, \qquad n = 3; \qquad (2,5)$$

$$\eta(x,t) = -\frac{4}{x}\sqrt{\frac{\omega}{2\pi}}\left[(\sqrt{2}+2)\sqrt[4]{2}\,e^{-\omega/2\sqrt{2}}\sin\left(\frac{\omega}{2\sqrt{2}} - \frac{\pi}{4}\right) + \right.$$

$$\left. +\sqrt[4]{8}\,e^{-\omega/2\sqrt{2}}\cos\left(\frac{\omega}{2\sqrt{2}} - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\cos\left(\frac{\omega}{4} - \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\sin\left(\frac{\omega}{4} - \frac{\pi}{4}\right)\right], \qquad n = 4; \qquad (2,6)$$

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[2] E. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Moscow, 1948.
[3] B. N. Rumyantsev, *Applied Mathematics and Mechanics*, **24**, issue 3 (1960).