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Abstract

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MATHEMATICS

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ON GROUPS WITH A k-REDUCIBLE BASIS

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The groups with a δ-basis and with a k-reducible basis introduced by V. A. Tartakovskii are of interest in view of the existence for them, respectively for $\delta < {}^1/_6$ and $k > 6$, of algorithms solving the identity problem*. The question arises of finding conditions, independent of the way in which the group is given, that are necessary or sufficient for the existence of a k-reducible basis with "good" k**. One such necessary condition is given in ([2]). In the present note other necessary conditions are given (Theorems 5 and 6), one of which absorbs the condition from ([2]). To obtain these conditions, Theorems 2, 3, and 4 are proved, which are also of independent interest. The question of the relationship between the notions of a k-reducible basis and a δ-basis is also investigated (Theorem 1).

Let G be a finitely defined group with defining relations $A_i = 1$ $(i = 1, \dots, n)$, where the words A_i are irreducible and externally irreducible (i.e. the first and last letters of A_i are not mutually inverse). The closure of the set $\{A_i\}$ with respect to the operations of cyclic permutation of letters and free inversion will be called the **basis** of the group, and the words belonging to the basis will be called **basis words**.

Let A be a basis word and B a segment of a basis word, where $A \doteq A'F,\ B \doteq F^{-1}B',\ F \neq \Lambda$***. The operation on a pair of words $\{A, B\}$, the result of which is the pair of words $\{A', B'\}$, will be called a **left reduction** of B by means of A. A **right reduction** of B by means of A is defined analogously.

If M, N are basis words and $N \doteq PQR$, then the left (right) reduction of Q by means of M will be called **simple** if $M^{-1} \neq QRP$ $(M^{-1} \neq RPQ)$. The simplicity of the reduction depends not only on Q and M, but also on N.

Further, if N is a basis word and $N \doteq RQR$, then an operation consisting in the successive application to Q of s simple reductions (some of which may be left and some right) by means of some basis words will be called an **s-fold simple reduction** of Q. If the word Q can be transformed into Λ by an s-fold simple reduction, we shall call it **s-reducible**.

Definition 1. Let k be a natural number. A basis is called **k-reducible** if: 1) for $t < k$ no basis word is t-reducible; 2) there exists a k-reducible basis word; 3) every basis word is s-reducible for some s.

* The first such algorithm was found by V. A. Tartakovskii ([16,]). Subsequently M. Greendlinger (in a work whose contents he presented at the seminar on mathematical logic at Moscow State University on 27 IV and 4 V 1958) indicated another, much simpler algorithm for groups with a δ-basis. For some related classes of groups the identity problem was solved by Britton ([4]) and Shik ([3]).

** P. S. Novikov drew my attention to this question, to whom I express my deep gratitude.

*** $\doteq$ denotes graphical equality, F^{-1} is the free inverse of F, Λ is the empty word.

Definition 1′. A basis satisfying conditions 1) and 2) of Definition 1 is called a **generalized k-reducible basis**.

Definition 1″. A basis is called **irreducible** if no basis word is s-reducible for any s.

We shall call the **length** a function $l(P)$, defined on the set of words in some alphabet and satisfying the conditions: 1) $l(P) \geq 0$; 2) $l(PQ) \leq l(P) + l(Q)$.

Definition 2. Let a length l be defined on the set of words in the alphabet of a finitely defined group G, and let $\delta > 0$ be a real number. A basis of the group is called a **δ-basis with respect to l** if, for any basis words M, N, from $M \not\doteq N^{-1}$, $M \underset{\circ}{=} M'F$, $N \underset{\circ}{=} F^{-1}N'$ it follows that $l(M') \geq l(M)(1-\delta)$, $l(N') \geq l(N)(1-\delta)$*.

Theorem 1. *Whatever the natural number k, in a finitely defined group with a generalized k-reducible or irreducible basis one can define a length such that the basis with respect to this length will be a $\frac{1}{k}$-basis. Conversely, for $\delta < \frac{1}{s}$, every δ-basis (with respect to any length) is either generalized k-reducible for some $k > s$, or irreducible.*

In what follows we use the notions of composition and product introduced by V. A. Tartakovskii ([1a]).

Theorem 2. *Every nonempty irreducible Dyck word of a finitely defined group G (i.e., a word equal to 1 in G) contains an occurrence of a nonempty word freely equal to a product of basis words**.*

A product of basis words is called **simple** if, under any method of reducing it, only simple reductions occur between its factors (cf. ([1a]), Ch. II, § 5).

Theorem 3. *Every nonempty irreducible Dyck word of a finitely defined group G contains an occurrence of a nonempty word freely equal to a simple product of basis words.*

Theorem 4. *If G is a finitely defined group with a generalized k-reducible basis, where $k \geq 6$, or with an irreducible basis, then for every nonempty irreducible Dyck word P of the group G there exist words F, G, A, B and a basis word M such that $P \underset{\circ}{=} AFB$, $M \underset{\circ}{=} FG$, and G is either empty or i-reducible, where $i \leq 3$**.**

Theorem 5. *A group with a k-reducible basis for $k \geq 6$ cannot be periodic.*

Theorem 6. *In a group with a k-reducible basis for $k \geq 8$, no nontrivial (i.e., not true in every group) identity relation can hold.*

Theorems 5 and 6 are also true for groups with a generalized k-reducible basis and with an irreducible basis, except for the following trivial cases: 1) cyclic groups with generators $a, b_1, \dots, b_n$ and relations $ab_1^{\beta_1} = 1, \dots, ab_n^{\beta_n} = 1$, to which $a^\alpha = 1$ may be added; 2) the group with generators a, b and relations $a^2 = 1$, $b^2 = 1$ (in it the identity $x^2y^2x^{-2}y^{-2} = 1$ holds).

For groups with an a-reducible basis in the sense of V. A. Tartakovskii' s original definition, Theorems 5 and 6 are likewise valid if the formal orders of all generators (([1a]), Ch. I, § 1) are infinite.

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* Definitions 1 and 2 are not equivalent to the corresponding definitions of V. A. Tartakovskii, but are very close to them.

** This theorem generalizes the theorem from ([1a]), Ch. I, § 4, proved for the case when each generator occurs in some basis word.

*** When the present paper had been written, I became aware of the work ([4b]), where a theorem very close to Theorem 4 is proved. The class of groups considered in ([4b]) is close to the class of groups with a δ-basis for $\delta < 1/6$, but neither contains the other.

If finite formal orders are present, these theorems can be proved with the lower bound for k increased respectively to 7 and to 10.

Let us also note the following generalization of Theorem 4:

Theorem $4'$**.** *Let every basis word B_i of the group G be represented in the form $B_i \doteq B_{i1} \dots B_{is_i}$, where $s_i \geqslant 6$, and let every product P of basis factors of the group G be freely equal to such a product Q of basis factors (not necessarily the same ones) that, under any reduction of Q, every simple cancellation between factors B_i and B_j annihilates no more than one B_{ik_i} and no more than one B_{jk_j} $(1 \leqslant k_i \leqslant s_i,\ 1 \leqslant k_j \leqslant s_j)$. Then every Dick word of the group G contains a segment of some basis word B_l having the form $B_{lt}B_{l,t+1}B_{l,t+2}$.*

The assertion of Theorem $4'$ remains true if the arbitrary product P is replaced by a simple one, while requiring in addition that Q also be simple.

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REFERENCES

1. V. A. Tartakovskii, a) *Mat. sbornik*, **25** (67), 1, 3 (1949); b) **25** (67), 2, 251 (1949); c) *Izv. AN SSSR, ser. matem.*, **13**, No. 6, 483 (1949).

2. A. V. Gladkii, DAN, **125**, No. 5 (1959).

3. H. Schiek, *Acta Math.*, **96**, 157 (1956).

4. J. L. Britton, a) *Proc. Glasgow Math. Assoc.*, **3**, 1, 45 (1956); b) **3**, 2, 68 (1957).

Note: Figure translations are in progress. See original paper for figures.