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Abstract

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STABILITY OF A CANTILEVER CYLINDRICAL SHELL WITH A REINFORCED EDGE UNDER EXTERNAL PRESSURE

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The question of the stability of a cantilever circular cylindrical shell is considered, for which the movable edge is reinforced by an elastic ring, under the action of an external pressure $q > 0$. For simplicity of exposition, the cross section of the ring is assumed rectangular, with base a and height H. The interaction of the ring and the shell is characterized by the forces $\overline{T}_1$, $\overline{S}$, $\overline{N}_1$ and the moments $\overline{M}_1$, $\overline{H}$, acting in the transverse section cutting the shell off from the ring. In the precritical state, among these force factors only $\overline{N}_1$ and $\overline{M}_1$ are nonzero. At loss of stability all of them, generally speaking, are nonzero and may be represented in the form of sums of the principal and additional values. Applying the indicated forces and moments in the corresponding directions to the shell and to the ring, one may consider them in isolation from one another.

Let us first turn to the shell. Starting from the complete linearized equilibrium equations, one can obtain for the additional displacements u, v, w of the shell the equations:

$$\Delta^4 u = \left(\nu \frac{\partial^3}{\partial \xi^3} - \frac{\partial^3}{\partial \xi \partial \varphi^2}\right) \Delta^2 w + (1+\nu)^2 \varepsilon \frac{\partial}{\partial \xi}\left[\frac{\partial^4}{\partial \varphi^4}\Delta^2 w + \frac{\partial^2}{\partial \xi^2} L_u^{(6)}(w) + \frac{\partial^6 w}{\partial \varphi^6} + \frac{\partial^2}{\partial \xi^2} L_u^{(4)}(w)\right] + \frac{qR}{Eh}\frac{\partial^3}{\partial \xi^3} L_0^{(4)}(w), \tag{1}$$

$$\Delta^4 v = \left[\frac{\partial^3}{\partial \varphi^3} + (2+\nu)\frac{\partial^3}{\partial \xi^2 \partial \varphi}\right] \Delta^2 w - (1-\nu^2)\varepsilon \left[\frac{\partial^5}{\partial \varphi^5}\Delta^2 w + \frac{\partial^2}{\partial \xi^2} L_v^{(7)}(w) + \frac{\partial^7 w}{\partial \varphi^7} + \frac{\partial^2}{\partial \xi^2} L_v^{(5)}(w)\right] + \frac{qR}{Eh}\frac{\partial^5}{\partial \xi^4 \partial \varphi} L_*^{(2)}(w), \tag{2}$$

$$\varepsilon\Delta^6 w + \frac{\partial^4}{\partial\xi^4}\Delta^2 w + \varepsilon\left[2\frac{\partial^6}{\partial\varphi^6}\Delta^2 w + \frac{\partial^8 w}{\partial\varphi^8} + \frac{\partial^2}{\partial\xi^2}L_1^{(6)}(w)\right] + \frac{qR}{Eh}\left[\left(\frac{\partial^2}{\partial\varphi^2}+1\right)\Delta^4 w + \frac{\partial^2}{\partial\xi^2}L_2^{(2)}(w)\right] = 0, \tag{3}$$

where $R\xi$ and $R\varphi$ are curvilinear coordinates on the middle surface of the shell in the axial and circumferential directions (at the fixed support $\xi = 0$; at the junction with the ring $\xi = \rho = l/R$; R and l are the radius and length of the shell); $\varepsilon = h^2/12R^2(1-\nu^2)$; h is the shell thickness; E and ν are the modulus of elasticity and Poisson' s ratio; $\Delta = \partial^2/\partial\xi^2 + \partial^2/\partial\varphi^2$; $L_u^{(6)}, L_u^{(4)}, \ldots$ are linear differential operators whose order coincides with the order of differentiation with respect to φ and is equal to the upper index. Assuming that in the case under consideration (as also for a shell with fixed edges)

$$\partial^2 u/\partial\xi^2 \ll \partial^2 u/\partial\varphi^2, \qquad \partial^2 v/\partial\xi^2 \ll \partial^2 v/\partial\varphi^2, \qquad \partial^2 w/\partial\xi^2 \ll \partial^2 w/\partial\varphi^2$$

(this condition, if the expressions adopted below for u, v, w are used, is equivalent to the inequality $\mu^2 \ll n^2$, which is consistent with the results of calculations), and bearing in mind that $\varepsilon \ll 1$, $qR/Eh = \sigma_2/E \ll 1$ (σ_2 is the circumferential stress in the shell), one may write instead of (1)–(3) *

$$\partial^2 u/\partial\varphi^2 = -\partial w/\partial\xi + (1+\nu)^2\varepsilon\,\partial^3 w/\partial\xi\,\partial\varphi^2, \tag{4}$$

$$\partial v/\partial\varphi = w - (1-\nu^2)\varepsilon\,\partial^2 w/\partial\varphi^2, \tag{5}$$

$$\varepsilon\left(\frac{\partial^8 w}{\partial\varphi^8} + 2\frac{\partial^6 w}{\partial\varphi^6} + \frac{\partial^4 w}{\partial\varphi^4}\right) + \frac{\partial^4 w}{\partial\xi^4} + \frac{qR}{Eh}\left(\frac{\partial^6 w}{\partial\varphi^6} + \frac{\partial^4 w}{\partial\varphi^4}\right) = 0. \tag{6}$$

If, using (1), (2), one forms the expression $\partial\Delta^4 u/\partial\varphi + \partial\Delta^4 v/\partial\xi$ and neglects in it, on the basis of the above inequalities, the corresponding terms, then, after reduction ** by $\partial^5/\partial\varphi^5$, we obtain

$$\frac{\partial^3}{\partial\varphi^3}\left(\frac{\partial u}{\partial\varphi} + \frac{\partial v}{\partial\xi}\right) = 2(1+\nu)\frac{\partial^3 w}{\partial\xi^3} + 2\nu(1+\nu)\varepsilon\frac{\partial^3}{\partial\xi\,\partial\varphi^2}\left(\frac{\partial^2 w}{\partial\varphi^2} + w\right). \tag{7}$$

Let us note that (7) cannot be derived from (4), (5) (the first term in the right-hand side, which is of essential importance below, would be absent). In deriving (7), the use of (4), (5) is illegitimate for the following reason. Equalities (4), (5) were obtained by neglecting in (1) and (2) the terms $\nu\partial^3\Delta^2 w/\partial\xi^3$ and $(2+\nu)\partial^3\Delta^2 w/\partial\xi^2\partial\varphi$ in comparison with the principal terms $-\partial^3\Delta^2 w/\partial\xi\,\partial\varphi^2$

and $\partial^3\Delta^2 w/\partial\varphi^3$. But these principal terms cancel each other in forming the expression $\partial\Delta^4 u/\partial\varphi + \partial\Delta^4 v/\partial\xi$.

Let us now turn to the equilibrium equations of the ring for its deformed state, given in ([2]). Of them only the following three are used:

$$dN/ds - N'\tau_1 + T\chi_1' + X = 0, \qquad dT/ds - N\chi_1' + N'\chi_1 + Z = 0,$$

$$dG'/ds - H\chi_1 + G\tau_1 + N + K' = 0, \tag{8}$$

where, in particular, $\chi_1' = R^{-1} + R^{-2}d^2\tilde{u}/d\varphi^2 + R^{-1}d\widetilde{w}/d\varphi$ ($\tilde{u}$ and $\widetilde{w}$ are the displacements of a point of the ring centerline in the radial and circumferential directions; for the remaining notation see ([2])).

At loss of stability the quantities $\tilde{u}, ...; T, ...; X, ...$ are written in the form $\tilde{u}_0 + \tilde{u}, ...; T_0 + T, ...; X_0 + X, ...$, where the first terms are the basic ("precritical") ones, and the second are additional ones. The change of the components of the external load $X, Z, K', ...$ upon loss of stability, depending on the change in the shape of the ring, is an essential feature of the present problem in comparison with the problem of stability of a ring under a fixed load. Fulfillment of the remaining three equilibrium equations of the ring is connected with satisfaction of the following conjugacy conditions: $u = \tilde{v}$, $R^{-1}w_\xi = -\beta$ at $\xi = \rho$, where u is the axial displacement of the shell, $\tilde{v}$ is the corresponding displacement of the ring, and β is the angle of rotation of the cross section of the ring. Here no attempt is made to take these conditions into account (just as boundary conditions for axial displacements and angles of rotation are usually not taken into account in studying the stability of a shell with fixed edges), and therefore the ring equilibrium conditions not given here are ignored. From (8) one obtains the following system of linearized equations for the additional displacements and force factors of the ring:

$$dN/d\varphi + T + T_0R^{-1}\left(d^2\tilde{u}/d\varphi^2 + d\widetilde{w}/d\varphi\right) + RX = 0,$$

$$dT/d\varphi - N + RZ = 0, \qquad dG'/d\varphi + RN + RK' = 0. \tag{9}$$

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* Without rigorous proof, equation (6) was obtained by A. V. Sachenkov ([1]).

** Reductions of this kind were also made in deriving (4), (5). They are legitimate, since the additional displacements must be periodic functions and will below be sought in trigonometric form.

In this case, in the precritical stressed state the ring is considered momentless, i.e., of the quantities $T_0, N_0, ...$, only T_0 is taken into account, while the others are set equal to zero. Such a simplifying assumption could have been avoided, but the analysis carried out showed that it has practically no effect on the results of the solution. Excluding the quantities T and N from (7) and using the relations

$$G' = EIR^{-2}\left(d^2\tilde{u}/d\varphi^2 + d\tilde{w}/d\varphi\right), \qquad T_0' = -kaRq$$

$$\left(I = \frac{1}{12}aH^3 \text{ is the moment of inertia of the cross section of the ring, } k = \text{const}\right),$$

we obtain

$$\frac{EI}{R^2}\frac{d}{d\varphi}\left(\frac{d^2}{d\varphi^2}+1\right)\left(\frac{d^2\tilde{u}}{d\varphi^2}+\frac{d\tilde{w}}{d\varphi}\right) + kaRq\frac{d}{d\varphi}\left(\frac{d^2\tilde{u}}{d\varphi^2}+\frac{d\tilde{w}}{d\varphi}\right) -$$

$$- R^2(dX/d\varphi - Z) + (d^2/d\varphi^2+1)RK' = 0, \tag{10}$$

where

$$X = -\overline{N}_1 = -N_1\big|_{\xi=\rho} = -\left[\frac{Eh^3}{12(1-\nu^2)R^3}\right]\left(\frac{\partial^3 w}{\partial\xi\,\partial\varphi^2} + \frac{\partial^2 v}{\partial\xi\,\partial\varphi}\right)_{\xi=\rho},$$

$$Z = -\overline{S} = -S\big|_{\xi=\rho} = -\left[\frac{Eh}{2(1+\nu)R}\right]\left(\frac{\partial u}{\partial\varphi} + \frac{\partial v}{\partial\xi}\right)_{\xi=\rho}, \tag{11}$$

$$K' = -\overline{H} = -H\big|_{\xi=\rho} = -\left[\frac{Eh^3}{12(1+\nu)R^2}\right]\left(\frac{\partial^2 w}{\partial\xi\,\partial\varphi} + \frac{\partial v}{\partial\xi}\right)_{\xi=\rho}.$$

(S, N_1 and H are the additional shearing force, transverse force, and twisting moment in the shell.) The value of k is obtained from the solution of the problem of the joint deformation of the shell and the ring in the precritical state. We have:

$$k = (1 + c_1 H/h + c_2)/(1 + c_1 + c_2),$$

where

$$c_1 = \frac{hR}{2\varkappa aH}\left(2 + \frac{6hR^3}{\varkappa^3 a^3 H}\right), \qquad c_2 = \frac{3hR^3}{\varkappa^3 a^3 H}\left(2 + \frac{\varkappa a}{R}\right),$$

$$\varkappa = \left[3(1-\nu^2)\right]^{1/4}(R/h)^{1/2}.$$

Since $\tilde{u} = w|_{\xi=\rho}$, $\tilde{w} = v|_{\xi=\rho}$, equation (10) can be written in the form

$$\left[\frac{EI}{R^2}\frac{\partial}{\partial\varphi}\left(\frac{\partial^2}{\partial\varphi^2}+1\right)\left(\frac{\partial^2 w}{\partial\varphi^2}+\frac{\partial v}{\partial\varphi}\right)+kaRq\frac{\partial}{\partial\varphi}\left(\frac{\partial^2 w}{\partial\varphi^2}+\frac{\partial v}{\partial\varphi}\right)\right]_{\xi=\rho}-$$

$$-R^2(\partial X/\partial\varphi - Z) + (\partial^2/\partial\varphi^2+1)RK' = 0. \qquad (12)$$

Multiplying (12) by $\partial^3/\partial\varphi^3$ and using (4), (5), (7), (11), after discarding small terms we shall have:

$$\left[\frac{H^3}{12hR^2}\frac{\partial^4}{\partial\varphi^4}\left(\frac{\partial^2}{\partial\varphi^2}+1\right)^2 w + \frac{kRq}{Eh}\frac{\partial^4}{\partial\varphi^4}\left(\frac{\partial^2}{\partial\varphi^2}+1\right)w-\right.$$

$$\left.-\frac{2-\nu}{12(1-\nu^2)}\frac{h^2}{Ra}\frac{\partial^5}{\partial\xi\,\partial\varphi^4}\left(\frac{\partial^2 w}{\partial\varphi^2}+w\right)-\frac{h^2}{12(1-\nu^2)Ra}\frac{\partial^3}{\partial\xi\,\partial\varphi^2}\left(\frac{\partial^2 w}{\partial\varphi^2}+w\right)-\frac{R}{a}\frac{\partial^3 w}{\partial\xi^3}\right]_{\xi=\rho}=0. \qquad (13)$$

(the last term in the square brackets of equality (13), which substantially affects the magnitude of the critical load, is due to the first term on the right-hand side of equality (7)). Put $w = C\sin\mu\xi\sin n\varphi$, thereby satisfying the condition $w = M_1 = 0$ at $\xi = 0$ (C is an arbitrary nonzero constant; μ is a constant whose value is determined in the course of the solution; n is a positive integer greater than unity; to the given w there correspond $u = A\cos\mu\xi\sin n\varphi$, $v = B\sin\mu\xi\cos n\varphi$). Then from (6) and (13) we obtain, respectively:

$$qR/Eh = \varepsilon(n^2-1) + \mu^4/n^4(n^2-1), \qquad (14)$$

$$\frac{qR}{Eh} = \frac{H^3(n^2-1)}{12khR^2} + \frac{h^2(2-\nu-n^{-2})}{12(1-\nu^2)kaR}\frac{\mu}{\operatorname{tg}\mu\rho} + \frac{R}{ka}\frac{\mu^3}{n^4(n^2-1)\operatorname{tg}\mu\rho}. \qquad (15)$$

From (14), (15), for the chosen n, the value μ and the corresponding eigenvalue of the pressure q are determined. The least of the eigenvalues q corresponding to the values $n = 2, 3, 4$ is the critical pressure q_{cr}. It can be found by constructing, for different values of n, graphs of the quantity qR/Eh as a function of μ according to formulas (14), (15). A pair of graphs constructed, respectively, according to formulas (14), (15) for one and the same value of n intersect at one point, which corresponds to the eigenvalue q. Constructing, for a series of values of n, pairs of the indicated graphs and determining their points of intersection, we find $q_{\text{cr}}R/Eh$ as the ordinate of the lowest of these points. In this case the variation of μ is bounded by the interval $0 < \mu < \bar{\mu} = \pi/\rho$, since for $\mu = \bar{\mu}$ the right-hand side of formula (15) tends to $-\infty$, and n must not exceed the number $\bar{n}$, equal to the critical value of n in the case of an absolutely rigid ring, i.e., when both edges of the shell are fixed in the radial direction (the corresponding value is $\mu = \bar{\mu}$). For a shell of "medium" length, $\bar{n}$ is equal to the integer part of the number $2.77\,N$, $N = (1-\nu^2)^{1/4}(\bar{R}/l)^{1/2}(R/h)^{1/4}$. Let us note that the values of μ and n obtained as a result of the calculations and corresponding to q_{cr} agree with the adopted assumption that $\mu^2 \ll n^2$.

If the stiffness of the ring is increased without bound, formally letting a or H tend to ∞, then satisfaction of equality (13) reduces to fulfillment of the condition $w|_{\xi=\rho} = 0$ (bearing in mind the expression for w indicated above), and this means that $\mu\rho = \pi$. Then at both ends of the shell the conditions $w = 0$, $M_1 = 0$ are satisfied. If the stiffness of the ring is decreased without bound, letting a tend to 0 and H to h, or without changing H, then satisfaction of equality (15), multiplied by a, which is equivalent to satisfaction of equality (13), reduces to fulfillment of the condition $\tan\mu\rho = \infty$, whence $\mu\rho = \pi/2$. Then, as it should be at the free edge, $S - H/\bar{R} = N_1 + R^{-1}\partial\tilde{H}/\partial\varphi = 0$ for $\xi = \rho$. In this case (when the ring is absent), for a shell of "medium"length (when $\varepsilon^{1/2} \ll (\pi R/l)^2 \ll \varepsilon^{-1/2}$) it is easy to find $q_{\text{cr}} = \underline{q}_{\text{cr}}$ directly, by determining the minimum with respect to n^2 of the right-hand side of formula (14), where we set $\mu = \pi/2\rho$ and neglect unity in comparison with n^2. As a result we obtain

$$\underline{q}_{\text{cr}} = \left[\frac{\pi E h^2}{3\sqrt{6}\,(1-\nu^2)^{1/4} R l}\right](h/R)^{1/2} \tag{16}$$

(i.e., $\underline{q}_{\text{cr}}$ is equal to half the critical pressure for the case when both ends of the shell are fixed), and the corresponding value $n = \underline{n}$, equal to the integer part of the number $(2.77/\sqrt{2})\,N$. It is clear that for a cantilever shell of "medium"length with a reinforcing ring at the movable edge one may practically determine q_{cr} by constructing the above-mentioned graphs in the following intervals of variation of μ and n: $\pi/2\rho \leqslant \mu < \pi/\rho$, $\underline{n} \leqslant n \leqslant \bar{n}$.

An experimental investigation was carried out of the stability of 20 cantilever shells with a free and with a reinforced edge. All the experimental results differ from the theoretical ones by less than 10%.

The obtained formula (16) for a shell with a free edge and its experimental verification refute the formula of N. A. Alfutov [3] (formula (4.9) for $n = 2$).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.