# PHYSICS

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## Abstract

## Full Text

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# ON THE QUESTION OF THE DEFORMATION OF THE ACCOMPANYING SPACE IN EINSTEIN' S THEORY OF GRAVITATION

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1. In Einstein' s theory of gravitation, the transition from homogeneous isotropic cosmological models to an anisotropic inhomogeneous universe leads to an increase in the variety of types of behavior of the accompanying (mass, or matter) space permitted by the equations of the gravitational field under certain physically natural requirements. In particular, the passage of the volume of any element of the accompanying space through a regular finite minimum becomes admissible not only for a cosmological constant $\Lambda > 0$, as in the case of homogeneous isotropic models under the requirements mentioned, but also for $\Lambda < 0$ and $\Lambda = 0$. Also admissible becomes a simultaneous combination of expansion of the volume of the accompanying space in one region with its contraction in another region, which is inconceivable in the case of homogeneous models.

Instead of the usual (direct) method, which presupposes the complete specification of the structure of the world (i.e., space-time) energy-momentum tensor $T_{\mu\nu}$ and the finding or investigation of the solutions corresponding to it, we shall apply a kind of semi-inverse method. Namely, specifying only the most general physical properties of $T_{\mu\nu}$, we shall at the same time impose the requirements we need on the solutions themselves and clarify the possibility of satisfying Einstein' s equations under these conditions. Proceeding in this way, we shall impose 4 relations on the world metric tensor $g_{\mu\nu}$, and 6 relations (partly also containing $g_{\mu\nu}$ and $\partial g_{\mu\nu}/\partial x^0$) on $T_{\mu\nu}$.

2. We shall use the equations of the law of gravitation and the equations of the law of energy and momentum following from them in the form

$$\left.\begin{aligned} G_{0\nu} - \frac{1}{2} g_{0\nu} G + \Lambda g_{0\nu} &= -\varkappa T_{0\nu}, \quad (1\alpha) \\ G_{ik} - \frac{1}{2} g_{ik} G + \Lambda g_{ik} &= -\varkappa T_{ik}, \quad (1\beta) \end{aligned}\right\} \tag{1}$$

$$(T_0^\nu)_\nu = 0, \qquad (T_i^\nu)_\nu = 0, \tag{2}$$

where $G_{\mu\nu}$ is the contracted world curvature tensor; $G = G^{\nu}_{\nu}$; $\varkappa$ is Einstein' s gravitational constant ($\varkappa = 8\pi\gamma/c^2$, $\gamma$ is Newton' s gravitational constant, $c$ is the fundamental speed), with Greek indices taking the values $0, 1, 2, 3$, and Latin ones only $1, 2, 3$. We shall also agree that all coordinates $x^\sigma$ are material; $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$; in a local Galilean system $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$. The quantities $g_{\mu\nu}$ and $T_{\mu\nu}$ will be assumed to be (material) holomorphic functions of the coordinates ( "analytic requirements" ).

3. Along with $g_{\mu\nu}$ we shall consider the set of quantities $\omega_\nu$ (not a vector!) and $y_{ik}$, defined by the equalities

$$\begin{gathered}(\omega_0)^2 = g_{00}, \qquad (\omega_0\omega_1)^2 = -\, g_{00}g_{11} + (g_{01})^2, \\ (\omega_0\omega_1\omega_2)^2 = gg^{33}, \qquad (\omega_0\omega_1\omega_2\omega_3)^2 = -\, g, \qquad y_{ik} = -\, g_{ik}.\end{gathered} \tag{3}$$

The necessary and sufficient conditions for the quantities $g_{\mu\nu}$ to correspond to the metric of the space-time continuum (see ([1]), p. 151; ([2]), p. 286) can, by virtue of (3), be represented in the form

$$(\omega_0)^2 > 0, \qquad (\omega_1)^2 > 0, \qquad (\omega_2)^2 > 0, \qquad (\omega_3)^2 > 0 \tag{4}$$

( "metric requirements" ). By arbitrarily prescribing the $\omega_\nu$ as positive holomorphic functions of the coordinates, we shall connect the quantities $g_{\mu\nu}$ with the first four relations of (3).

Transformations $\bar{x}^\mu = \bar{x}^\mu(x^0, x^1, x^2, x^3)$ for which $\bar{\omega}_\mu$ do not depend on $y_{ik}$ satisfy the condition $\partial\bar{x}^\mu/\partial x^\nu = 0$, $\mu > \nu$; then also $\partial x^\nu/\partial\bar{x}^\mu = 0$, $\nu > \mu$. In this case

$$\omega_\nu = (\partial\bar{x}^\mu/\partial x^\nu)\, \bar{\omega}_\sigma, \quad \mu = \nu = \sigma.$$

We shall use the definition, adopted in ([3–5]), of a space with chronometrically invariant (ch. i.) metric tensors

$$h_{ik} = -g_{ik} + g_{0i}g_{0k}/g_{00}$$

and

$$h^{ik} = -g^{ik},$$

and with fundamental determinant

$$h = -g/g_{00}$$

(coinciding with those adopted in ([1]), p. 248, and ([2]), p. 235), as well as other ch. i. quantities and ch. i. operators (marking the latter by asterisks). Speaking of three-dimensional tensors and other quantities and images of three-dimensional geometry, we shall omit mention of the number of dimensions.

For the ch. i. time $\tau_0$ of a stationary point we have

$$c\, d\tau_0 = \omega_0\, dx^0.$$

Further,

$$(\omega_1)^2 = h_{11}, \qquad (\omega_1\omega_2)^2 = hh^{33}, \qquad (\omega_1\omega_2\omega_3)^2 = h.$$

Consequently, prescribing the functions $\omega_\nu$ includes the choice (for any $x^i$ and $dx^i$) of the dependence on $\tau_0$ of the ch. i. quantities: 1) the length

$$dL_1 = \omega_1\, dx^1$$

of an element of the line $x^2 = \text{const}$, $x^3 = \text{const}$; 2) the area

$$dS_{12} = \omega_1\omega_2\, dx^1 dx^2$$

of an element of the surface $x^3 = \text{const}$, bounded by coordinate lines; 3) the volume

$$dV = \omega_1\omega_2\omega_3\, dx^1\, dx^2\, dx^3$$

of an element of space bounded by coordinate surfaces, and also the quantity

$$R = a\sqrt[3]{\omega_1\omega_2\omega_3}, \quad a = a(x^1, x^2, x^3) > 0, \quad \partial a/\partial x^0 = 0.$$

A discrepancy between the signs of $\partial\omega_1/\partial x^0$, $\partial(\omega_1\omega_2)/\partial x^0$, and $\partial(\omega_1\omega_2\omega_3)/\partial x^0$ is a sufficient indication of compression of space in some directions and expansion in others.

Considering $y_{ik}$ as a covariant metric tensor of the spatial section $x^0 = \text{const}$ ([4,5]), we shall find the fundamental determinant

$$y = |y_{ik}|$$

and the contravariant metric tensor of the section, which we denote by $z^{ik}$ (and not by $y^{ik}$, as in ([4,5])), so that

$$y_{ij}z^{kj} = \delta_i^k,$$

where $\delta_i^k$ is the Kronecker tensor. Then

$$\sqrt{g_{00}} = \omega_0, \qquad \frac{g_{01}}{\sqrt{g_{00}}} = \Omega_1, \qquad \frac{g_{02}}{\sqrt{g_{00}}} = \frac{y_{12}}{y_{11}}\Omega_1 - \frac{\omega_1}{y_{11}}\Omega_2, \tag{5}$$

$$\frac{g_{03}}{\sqrt{g_{00}}} = \frac{y_{13}}{y_{11}}\Omega_1 + \frac{z^{23}\omega_1}{y_{11}z^{33}}\Omega_2 - \frac{\omega_1\omega_2}{yz^{33}}\Omega_3, \qquad g_{ik} = -y_{ik},$$

where

$$\Omega_1 = k\sqrt{(\omega_1)^2 - y_{11}}, \qquad \Omega_2 = l\sqrt{(\omega_2)^2 y_{11} - yz^{33}}, \tag{6}$$

$$\Omega_3 = m\sqrt{y[(\omega_3)^2 z^{33} - 1]}, \qquad k^2 = l^2 = m^2 = 1.$$

4. We shall regard matter as a medium free of non-gravitational and non-inertial mass forces, and shall neglect the energy flux relative to it and the second viscosity. Having prescribed holomorphic functions $K_1, K_2, K_3$, we shall connect $T_{\mu\nu}$ by six relations (7)–(8):

$$J^i = 0; \qquad \rho = K_1(p); \qquad D_{il}\varepsilon^{il} = \rho K_2(h_{ik}, D_{ik}) \geq 0; \tag{7}$$

$$K_3(\rho, \varepsilon_{ik}, h_{ik}, D_{ik}) = 0. \tag{8}$$

The mass-flux density vector $J^i$, the mass density $\rho$, the pressure $p$, and the viscous stress tensor $\varepsilon^{ik}$ are defined by the equalities $T_{00} = \rho g_{00}$;

$cT_0^i = j^i\sqrt{g_{00}}$; $c^2T_l^{ik} = ph^{ik} - \varepsilon^{ik}$, $\varepsilon_i^i = 0$. For the tensor of deformation velocities of space $D_{ik}$ we have: $D_{ik} = \dfrac{c}{2}{}^*\partial h_{ik}/\partial x^0$. The equality of $J^i$ to zero expresses 3 conditions expressing the fact that the coordinate systems used accompany the mass, consequently (in our case) also the medium ( "kinematic requirements" ). The equality containing $K_1$ is the equation of state. Besides $\rho$ and $p$, it may also include other quantities, which must be connected with the other quantities entering into (1), (2), or (7). The relation containing $K_2$ expresses the physical requirement—the condition of increase (nondecrease) of entropy owing to viscosity. One can choose $K_1$ so that, by virtue of (2), with proper initial conditions and with $J^i = 0$, the following collection of physical requirements is also fulfilled: $\rho c^2 > 3p \geq 0$; when $D_{jl}\varepsilon^{jl} = 0$, the signs of $\partial\rho/\partial x^0$ and $\partial p/\partial x^0$ are opposite to the sign of $D \left(= D_i^i = c\,{}^*\partial \ln\sqrt{h}/\partial x^0\right)$; $\rho$ and $p$ are finite for $h \neq 0$. As an (artificial) example illustrating the possibility of such a choice, we give the equation $\rho c^2 = np + b/\sqrt{h}$, $3 \leq n = \text{const}$, $b = b(x^1, x^2, x^3) > 0$, $\partial b/\partial x^0 = 0$, in conjunction with the initial condition for the first of equations (2): $p = \varphi > 0$ for $x^0 = x_0^0$. Thus, the adopted kinematic and physical requirements impose on $T_{\mu\nu}$ 5 restrictions (7). The function $K_3$, solved with respect to the quantities $\varepsilon_{ik}$, must be specified additionally: relation (8), together with the last of relations (7), makes it possible to express 2 of the 5 essential components of the tensor $\varepsilon_{ik}$ through the remaining 3, which we shall denote by $q_{ik}$.

With a proper choice of $q_{ik}$ from $\varepsilon_{ik}$, and for $g_{0i} \neq 0$, equations (2), transformed with the aid of (7)–(8), are equations of first order in $x^0$ with respect to $p$ and $q_{ik}$, reducible to normal form, and containing also $g_{\mu\nu}$ and $\partial g_{\mu\nu}/\partial x^\sigma$.

5. Using (5)–(6) and (7)–(8), let us express the 20 functions $g_{\mu\nu}, T_{\mu\nu}$ through the 14 functions $\omega_\nu, y_{ik}, q_{ik}, p$, and in this way transform (1) and (2). In some world region $Q_\omega$ let the functions $\omega_\nu$ be prescribed. Then (2) and (1$\beta$) constitute a system of $4 + 6$ equations ($\Sigma$) with respect to $p, q_{ik}, y_{ik}$. Let $O(x_0^0, x_0^1, x_0^2, x_0^3)$ be a world point (the point $P(x_0^1, x_0^2, x_0^3)$ of space at $x^0 = x_0^0$) in the region $Q_\omega$. Prescribe the initial conditions: for $x^0 = x_0^0$, $p = \varphi(x^1, x^2, x^3) > 0$, $q_{ik} =$

$\chi_{ik}(x^1, x^2, x^3)$, $y_{ik} = \psi_{ik}(x^1, x^2, x^3)$, $\partial y_{ik}/\partial x^0 = \Psi_{ik}(x^1, x^2, x^3)$. The choice of the 16 functions $\varphi, \chi_{ik}, \psi_{ik}, \Psi_{ik}$, holomorphic in neighborhoods of the point $P$, we subject to the conditions (which is always possible): a) the right-hand sides of the equations $\Sigma$ reduced to normal form, considered as functions of 104 arguments: $x^\sigma$ (through $\omega_\nu$), $p, q_{ik}, y_{ik}, \partial y_{ik}/\partial x^0, \partial p/\partial x^j, \partial q_{ik}/\partial x^j, \partial y_{ik}/\partial x^j, \partial^2 y_{ik}/\partial x^j \partial x^0, \partial^2 y_{ik}/\partial x^l \partial x^j$, must be real and holomorphic in the region (neighborhoods) of the system of their values equal, respectively, to $x_0^\sigma$ and to the values of the quantities $\varphi, \chi_{ik}, \psi_{ik}, \Psi_{ik}, \partial\varphi/\partial x^j, \partial\chi_{ik}/\partial x^j, \partial\psi_{ik}/\partial x^j, \partial\Psi_{ik}/\partial x^j, \partial^2\psi_{ik}/\partial x^l \partial x^j$ at $P$; b) the 4 equations ($1\alpha$) must be satisfied for $x^0 = x_0^0$ by virtue of the initial data and ($1\beta$). If a) is fulfilled, there exists in some world region containing $O$ a system of holomorphic functions $p, q_{ik}, y_{ik}$ satisfying the equations $\Sigma$ and the initial conditions. If b) is fulfilled, this system of functions also satisfies the equations ($1\alpha$) in some world region containing $O$ (cf. ([6]), p. 319)*. Using (5)–(6) and (7)–(8), from the prescribed $\omega_\nu$ and the obtained $y_{ik}, q_{ik}, p$ we find functions $g_{\mu\nu}, T_{\mu\nu}$ satisfying, in some world region $Q_0$ containing $O$, equations (1)

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* For this it is sufficient for us that equations ($1\beta$) be satisfied in some world region containing $O$, and not "in the whole space," as is said in the cited passage (in the original, everywhere). There it is also erroneously stated that "the remaining 4 equations of gravitation do not contain second derivatives with respect to the time coordinate." It should be stated that Einstein' s equations contain second derivatives with respect to this coordinate only of the 6 spatial components of the world metric tensor.

and all the accepted (analytic, metric, kinematic, and physical—see §§ 2-4) requirements. Thus, for any quadruple of functions $\omega_\nu$, positive and holomorphic in the world domain $Q_\omega$, for any choice of the point $O$ of this domain (and hence for any behavior of the quantities $\omega_\nu$ in neighborhoods of $O$), and for any initial data (satisfying conditions a) and b)), there exists a world domain $Q_0$ depending on them (containing $O$), in which a solution of system (1) is defined that satisfies all the accepted requirements. What has been said is valid for any $\Lambda$, including $\Lambda = 0$.

In view of the analytic continuation of the solution, one may, in particular, make use of the coordinate transformations indicated in § 3. They make it possible to shift, within $Q_\omega$, the domain of variation of $x^\sigma$, in which the right-hand sides of the equations $\Sigma$, reduced to normal form, remain real holomorphic functions of their arguments.

6. The general conclusion of § 5 concerning the existence of solutions of system (1) (satisfying all the accepted requirements) for any behavior of the quantities $\omega_\nu$ in neighborhoods of a given world point $O$ applies, of course, also to the case of contraction of space in some directions while it expands in others (see § 3), and to the cases considered below.

1) In neighborhoods of $O$ draw through it an arbitrary section containing timelike lines,

$$f_t(x^0, x^1, x^2, x^3) = 0,$$

It is a surface moving with the variation of $x^0$ and passing through $P$ when $x^0 = x^0_0$. Suppose that at all world points of the section

$$\partial(\omega_1\omega_2\omega_3)/\partial x^0 = 0, \quad \partial^2(\omega_1\omega_2\omega_3)/(\partial x^0)^2 = 0, \quad \partial^2(\omega_1\omega_2\omega_3)/\partial x^i \partial x^0 \neq 0$$

(for the $x^i$ entering into the equation of the section), and, consequently,

$$D = 0, \quad {}^*\partial D/\partial x^i \neq 0$$

(for the same $x^i$). Obviously, the indicated surface serves as the boundary between the region of expansion and the region of contraction of the volume of space, and the simultaneity of expansion and contraction is absolute.

2) In neighborhoods of $O$ draw through it an arbitrary spacelike section

$$f_s(x^0, x^1, x^2, x^3) = 0.$$

Suppose that at all its world points

$$\partial(\omega_1\omega_2\omega_3)/\partial x^0 = 0, \quad \partial^2(\omega_1\omega_2\omega_3)/(\partial x^0)^2 > 0$$

and, consequently,

$$D = 0, \quad {}^*\partial D/\partial x^0 > 0.$$

Obviously, the volume of each element of space (at $P$ and in its neighborhoods) passes through a regular finite minimum at the value of $x^0$ connected with the values of the coordinates $x^1, x^2, x^3$ of the element by the equation of the section.

The character of the behavior of $R$ coincides with the character of the behavior of the volume.

The necessity of absolute rotation or of a negative physical divergence of the gravitational-inertial force for the realization of a regular minimum of $R$ when $\Lambda \leqslant 0$ is evident from what was said in ([3,5]). The possibility of removing the singularity at the beginning of expansion in the case of homogeneous rotating models was indicated conjecturally in ([7]).

By an appropriate specification of $\omega_0$ and $\omega_1\omega_2\omega_3$, one can obtain in $Q_\omega$ any dependence of $R$ on $\tau_0$, in particular one realizing type $O_2$ (oscillations of the second kind, i.e., between regular extrema). The question of the possibility, in this case (and with all the accepted requirements being fulfilled), of satisfying system (1) throughout the entire domain of interest to us reduces to the question of the possibilities of sufficiently enlarging the domain $Q_0$ for a prescribed type of behavior of $R$.

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*Note: Figure translations are in progress. See original paper for figures.*