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Abstract

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ON THE NUMERICAL INTEGRATION OF THE EQUATIONS OF THE LAMINAR BOUNDARY LAYER

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Below a method is given for the exact integration of the boundary-layer equations with arbitrary boundary conditions. The dynamic boundary layer in plane steady flows of an incompressible gas is considered mainly, although the fundamental basis of the method is readily generalized to the general case of laminar flows of a compressible gas.

In deriving the boundary-layer equations it is quite immaterial which surface inside the layer is taken as the initial surface for measuring the transverse coordinate y. Mathematically this is expressed by the invariance property of the boundary-layer equations with respect to a translation transformation of the transverse coordinate. In established practice, the surface of the body being flowed around is taken as the initial surface $y = 0$ in the boundary layer. It is then assumed that the surface of the body in an external inviscid flow coincides with the surface of the body in a viscous flow. As a result, the asymptotic condition for the transverse component of velocity (and, in the presence of a vortical external flow, also for the longitudinal velocity components) cannot be exactly satisfied, which leads to a disturbance of the inviscid flow far from the body. Another approach is possible, however, one based on the exact satisfaction of all asymptotic conditions. As a result the viscous and inviscid flows far from the body coincide completely, but the surface of the body $y = y_w$ in the viscous flow turns out to be displaced from the surface of the body $y = 0$ in the inviscid flow by the displacement thickness $\delta = -y_w$. This approach appears more justified. In particular, allowance for the inverse influence of the boundary layer on the external inviscid flow becomes immediately evident; for this it is sufficient to solve the problem of the flow of an inviscid stream past a body whose surface is displaced by the displacement thickness. In what follows, as the initial surface $y = 0$ in the boundary layer we shall always take the displacement surface, i.e., the surface of the body in the inviscid flow.

Let us give the expression for the displacement thickness of the substance of a body in a plane flow of a viscous incompressible gas. In the neighborhood of the surface of the body $y = 0$, flowed around by an irrotational stream of an inviscid gas, we have, to within small quantities of order y^2,

$$u_e(x,y) = u_e(x,0); \qquad v_e(x,y) = -u_e'(x,0)y, \tag{1}$$

where the values of y are of the order of the boundary-layer thickness: $y \sim \sqrt{\nu}$. Using the continuity equation for the differences $u_e - u,\ v_e - v$ between the velocity components of the inviscid and viscous flows, and taking into account that far from the body these differences tend to zero, we find, with allowance for (1),

$$-u_e y_w = \int_{y_w}^{\infty} (u_e - u)\, dy + \int_0^x V_w\, dx; \qquad V_w = v_w - u_w y_w', \tag{2}$$

where V_w is the projection of the velocity vector onto the normal to the body surface, $d = y_w(x)$. Formula (2) has a simple physical meaning and characterizes the displacement of the boundary layer caused both by the retardation of particles and by the penetration of an additional mass of gas.

Let us now note the following result to which exact satisfaction of the asymptotic requirements leads. It is precisely under these conditions that, far from the body, the boundary-layer equations pass over into linear heat-conduction equations, whose form depends neither on the unknown quantities nor on the boundary conditions at the body surface. This circumstance makes it possible to formulate external boundary conditions for a **finite** region of the boundary layer **independently** of the boundary conditions on the body surface. Thus the principal difficulty in integrating the boundary-layer equations is eliminated, namely that the solution has to be considered in an infinite region.

Passing to the formulation of the external boundary conditions, we indicate the transformation

$$\tilde{x} = \int_0^x \frac{\nu}{c^2} u_e\, dx, \qquad \tilde{y} = \frac{u_e}{c} y; \qquad u = u_e \tilde{u}; \qquad v = c\left(\tilde{v}\frac{d\tilde{x}}{dx} - \tilde{u}\frac{\partial \tilde{y}}{\partial x}\right); \qquad c = \text{const},$$

which leads to equations for a fictitious incompressible flow $\tilde{u}, \tilde{v}$ in the plane $\tilde{x}, \tilde{y}$ with the simplest asymptotic conditions: $\tilde{u}_e = 1$, $\tilde{v}_e = 0$. From these conditions it follows that

$$\tilde{\psi} = \tilde{y} + \int_{\tilde{y}}^{\infty} (1 - \tilde{u})\, d\tilde{y}; \qquad \tilde{\psi} = \int_0^{\tilde{x}} \tilde{V}\, d\tilde{x} \quad (\tilde{V} = \tilde{v} - \tilde{u}\tilde{y}'(x)). \tag{3}$$

The first expression in (3) replaces the continuity equation, expressing explicitly the stream function $\tilde{\psi}$ through the sought function $\tilde{u}$. The second expression plays a role only in determining the boundary values for $\tilde{\psi}$ on the body surface

$\tilde{y} = \tilde{y}_w(x)$. In the coordinates $\tilde{x}, \tilde{y}$ the equation for the dynamic boundary layer far from the body takes the simplest form

$$\frac{\partial u_e^2 w}{\partial \tilde{x}} = \frac{\partial^2 u_e^2 w}{\partial \tilde{y}^2}; \qquad w = 1 - \tilde{u}. \tag{4}$$

In what follows we shall use the parabolic coordinates ξ, η:

$$\xi = \sqrt{2\tilde{x}}, \quad \xi\eta = y \; (\xi \tilde{f} = \tilde{\psi}).$$

Their introduction removes the singularity at the point $\tilde{x} = 0$: on the line $\xi = 0$ the boundary-layer equations always reduce to ordinary ones. In the coordinates ξ, η, the set of solutions w of equation (4) that vanish at infinity can be represented by the following integro-differential equation

$$\sqrt{\frac{\pi}{2}}\, w(\xi, \eta) = \int_0^\xi \frac{u_e^2(\xi_0)}{u_e^2(\xi)} \left[w(\xi_0, \eta) \frac{dt}{d\xi_0} + \frac{\tau(\xi_0, \eta)}{S} \right] e^{-t^2/2}\, d\xi_0; \qquad \tau = -\frac{\partial w}{\partial \eta}; \tag{5}$$

$$S = \sqrt{\xi^2 - \xi_0^2}; \qquad t = \eta \sqrt{\frac{\xi - \xi_0}{\xi + \xi_0}}; \qquad \frac{dt}{d\xi_0} = -\frac{\xi\eta}{S(\xi + \xi_0)}.$$

The function w satisfying equation (5) is at the same time a solution of the boundary-layer equations for values of η so large that quantities of order w^2 may be neglected in comparison with quantities of order w. To estimate such values of η, one may use

asymptotic formula

$$\frac{w(\xi, \eta)}{w(0, 0)} = \sqrt{\frac{2}{\pi} \frac{B^2 e^{2\beta_0}}{u_e^2(\xi)} \frac{2^{\beta_0} \Gamma(1 + 2\beta_0)}{\eta^{1+2\beta_0}}}\, e^{-\eta^2/2} \qquad \left(\beta(\xi) = \frac{\xi}{u_e} \frac{du_e}{d\xi};\ \beta_0 = \beta(0) \right),$$

obtained under the assumption that, in a small neighborhood of the point $\xi = 0$, $u_e(\xi) = B\xi^{\beta_0}$ ($B = \text{const}$).

Let us now fix some asymptotic solution, prescribing, for example, the values of τ on some sufficiently distant line $\eta = \eta_e$. Using this solution as the initial condition for the Cauchy problem, one can construct the solution of the boundary-layer equations in the whole region. The position of the body surface $\eta = \eta_w(\xi)$ is found from the condition $\tilde{u}(\xi, \eta_w) = \tilde{u}_w$. In this case definite boundary values for $\tilde{V}_w$ are obtained. If in the constructed solution one takes any other line $\eta = \eta_w(\xi)$ as the body surface, then boundary values are determined both for $\tilde{u}_w$ and for $\tilde{V}_w$. Thus, a fixed asymptotic solution determines not one, but a certain family of solutions of the boundary-layer equations; the whole family of solutions w of the boundary-layer equations that vanish at infinity, for all possible boundary conditions on the body surface, depends on only one arbitrary function of ξ.

Suppose now that all boundary conditions on the body surface are prescribed. Then equation (5) can play the role of an external boundary condition for a finite region. The solution of the corresponding boundary-value problem is substantially facilitated by the fact that the boundary-layer equations are equations of parabolic type. Consequently, the problem reduces to determining the unknown quantities on the characteristic $\xi = \xi_k + \Delta\xi$, under the condition that on the neighboring characteristic $\xi = \xi_k$ everything is known. The boundary conditions are also known for $\eta = \eta_w$ and for $\eta = \eta_e$. At the same time, on the initial characteristic $\xi = 0$, the boundary-layer equations reduce to ordinary equations. The solution of such an elementary problem is found as the solution of the Cauchy problem with initial condition $\tau(\eta_e) = \tau_e$. Since the value of τ_e is not known in advance, the first solution found will differ from the desired one. For small $\Delta\xi$ this difference will be small, which makes it possible to set up auxiliary linear equations determining a small correction to the solution found, with accuracy up to its square.

In the numerical solution of the formulated problem, the boundary-layer equations should be used in integral form

$$\tau = w\frac{d\xi f}{d\xi} + \frac{d\xi\vartheta}{d\xi} + \beta(\theta + \vartheta); \qquad f = \eta + \theta;$$

$$\theta = \int_\eta^\infty w\, d\eta; \qquad \vartheta = \int_\eta^\infty \widetilde{w}\, d\eta, \tag{6}$$

where the derivatives with respect to ξ may be taken along an arbitrary line $\eta = \eta(\xi)$. The dimensionless friction τ is determined by numerical differentiation with respect to ξ; therefore, in accordance with the formula

$$\frac{1}{2}\left[\frac{d\xi a}{d\xi} + \left(\frac{d\xi a}{d\xi}\right)^{(-)}\right] = \frac{\xi a - (\xi a)^{(-)}}{\Delta\xi} + O\left[(\Delta\xi)^2\right], \qquad \left(\frac{d\xi a}{d\xi}\right)_{\xi=0} = a, \tag{7}$$

the interval $\Delta\xi$ must be very small. But then, as follows from (6) and (7), the integral quantities f and ϑ must be computed with considerably greater accuracy than τ. Such accuracy can be ensured by a proper choice of numerical-integration formulas and of the size of the interval $\Delta\eta$.

As already mentioned, the solution of equations (6) on the characteristic $\xi = \xi_k + \Delta\xi$ is found as the solution of a Cauchy problem with initial conditions at the point $\eta = \eta_e$. At this point the asymptotic condition (5) may be approximately represented in the form

$$c_1 w_e + c_2 \tau_e = c_3, \tag{8}$$

where the quantities c_i have been computed in advance. Assigning the value of τ_e (for example, $\tau_e = \tau_e^{(-)}$), we determine from (8) the value of w_e. The value of θ_f ($\vartheta_e = \theta_e$) is determined from (6), where one should set $f = \eta_e$. Moving downward to smaller values of η, we determine the entire solution. The values η_w and f_w are found from the conditions $\tilde{u}(\eta_w) = u_w$, $f_w = \eta_w + \theta_w$. Since the value f_w is prescribed in advance, it is necessary to find the corresponding small correction to the solution obtained. For this purpose the sought solution must be represented in the form

$$w = w_0 + \varepsilon w_1; \qquad \tau = \tau_0 + \varepsilon \tau_1; \qquad \theta = \theta_0 + \varepsilon \theta_1; \ldots \qquad (\varepsilon = \text{const}),$$

where $w_0, \tau_0, \ldots$ is the solution found. It follows from (8) that the correction functions w_1 and τ_1 must satisfy the condition $c_1 w_{1e} + c_2 \tau_{1e} = 0$. The constant ε drops out of this condition because of the linearity of relation (8). In an analogous way, but (owing to the nonlinearity of the original equations) already to accuracy up to quantities of order ε^2, the quantity ε drops out of all the linear, taken together homogeneous, equations that are obtained for determining the correction solution. The corresponding initial value τ_{1e} may be arbitrary. The constant ε is found with account taken of the prescribed values for $\tilde{u}_w$ and f_w.

When advancing in ξ, it may prove necessary to change the value of η_e. For this one should use an outer condition of type (5), written on an arbitrary line $\eta = \eta_e(\xi)$. Such a condition, as well as an explicit expression for the solution for values $\eta > \eta_e$, is not difficult to obtain in the usual way, using the fundamental solution of the heat-conduction equation (1).

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REFERENCES

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Note: Figure translations are in progress. See original paper for figures.