MATHEMATICS

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Abstract

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MATHEMATICS

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ON EXPANSION IN EIGENFUNCTIONS OF A NON-SELF-ADJOINT DIFFERENTIAL OPERATOR OF ARBITRARY EVEN ORDER ON THE HALF-AXIS $[0, \infty)$

(Presented by Academician A. N. Kolmogorov on 21 I 1960)

The present work is an extension of the results obtained in ([1]) to the case of operators of arbitrary even order. The work uses methods first applied in ([1]).

Consider the differential expression:

$$l(y) = y^{(2n)} + p_2(x)y^{(2n-2)} + p_3(x)y^{(2n-3)} + \dots + p_{2n}(x)y, \tag{1}$$

where $p_k(x),\ k = 2, 3, \dots, 2n$, are complex-valued functions summable on the interval $[0, \infty)$. Denote by D the set of all functions $y(x) \in L^2(0, \infty)$ such that: 1) the derivatives $y^{(\nu)}(x)$, $\nu = 1, 2, \dots, 2n-1$, exist and are absolutely continuous in every finite interval $[0, b]$, $b > 0$; 2) $l(y) \in L^2(0, \infty)$.

Denote by D_α the set of all functions $y(x) \in D$ satisfying the boundary conditions

$$u_\nu(y) = \alpha_{\nu 0}y(0) + \alpha_{\nu 1}y'(0) + \dots + \alpha_{\nu,\, 2n-1}y^{(2n-1)}(0) = 0, \quad \nu = 1, 2, \dots, n, \tag{2}$$

where $\alpha_{\nu k}$ are complex numbers.

Define the operator L_α as follows: its domain of definition is D_α, and for $y \in D_\alpha$

$$L_\alpha y = l(y). \tag{3}$$

The operator L_α^*, adjoint to L_α, is constructed analogously for the differential expression adjoint to (1),

$$l^*(z) = z^{(2n)} + (\bar{p}_2 z)^{(2n-2)} - (\bar{p}_3 z)^{(2n-3)} + \dots + \bar{p}_{2n} z \tag{4}$$

and for the boundary conditions adjoint to (2),

$$v_\nu(z) = \beta_{\nu 0} z(0) + \beta_{\nu 1} z'(0) + \dots + \beta_{\nu,\, 2n-1} z^{(2n-1)}(0) = 0, \quad \nu = 1, 2, \dots, n. \quad (5)$$

Put $\rho^{2n} = -\lambda$. Let $\omega_1, \dots, \omega_{2n}$ be the roots of degree $2n$ of -1; divide the complex ρ-plane into $2n$ equal sectors S_k, $k = 0, 1, \dots, 2n-1$, defined by the inequality

$$\frac{k\pi}{n} < \arg \rho < \frac{(k+1)\pi}{n}.$$

In each sector S_k one can choose such an arrangement of the numbers $\omega_1, \dots, \omega_{2n}$ that for $\rho \in S_k$

$$\mathrm{Re}(\rho\omega_1) \leq \mathrm{Re}(\rho\omega_2) \leq \dots \leq \mathrm{Re}(\rho\omega_{2n})$$

(see ([2])). Denote by T_k and T_{k-1} the boundaries of the sector S_k. Suppose that the functions $p_k(x)$ satisfy the additional condition

$$e^{\varepsilon x} |p_k(x)| \leq c_k; \quad (6)$$

c_k are constants, ε_2 is a certain fixed number. It can be shown that the equation $l(y) = \lambda y$ has linearly independent solutions $y_k(x, \rho)$, $k = 1, \dots, 2n$, holomorphic with respect to ρ for $\rho \in S_k$ and satisfying the asymptotic conditions:

as $x \to +\infty$,

$$y_k^{(\nu)}(x, \rho) = \rho^\nu e^{\rho\omega_k x} \left[\omega_k^\nu + O(1)\right] \quad (7)$$

uniformly with respect to $\rho \in S_k$;

as $\rho \to \infty$,

$$y_k^{(\nu)}(x, \rho) = \rho^\nu e^{\rho\omega_k x} \left[\omega_k^\nu + O\left(\frac{1}{\rho}\right)\right] \quad (8)$$

uniformly with respect to $x \in [0, \infty)$.

In the domain $\widetilde{S}_k$, defined by the relation

$$|\mathrm{Re}(\rho\omega_n)| \leq \varepsilon_1, \qquad 0 < \varepsilon_1 < \varepsilon_2, \qquad \rho \in S_k \text{ or } S_{k+1},$$

the solution y_{n+1} is replaced by $\hat{y}_{n+1}$, which also satisfies conditions (7), (8).

Denote

$$A(\rho) = \begin{vmatrix} u_1(y_1) \dots u_1(y_{n-1}) & u_1(y_n) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ u_n(y_1) \dots u_n(y_{n-1}) & u_n(y_n) \end{vmatrix}, \qquad \widetilde{A}(\rho) = \begin{vmatrix} u_1(y_1) \dots u_1(y_{n-1}) & u_1(\hat{y}_{n+1}) \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ u_n(y_1) \dots u_n(y_{n-1}) & u_n(\hat{y}_{n+1}) \end{vmatrix}.$$

For simplicity we shall assume that

$$A(\rho) \neq 0, \qquad \widetilde{A}(\rho) \neq 0 \tag{9}$$

for $\rho \in T_k$, and that the eigenvalues of the operator L_α are simple.

Theorem 1. The spectrum of the operator L_α is continuous for even n on the positive semi-axis (for odd n, respectively, on the negative semi-axis) and is discrete in the entire remaining complex λ-plane. The eigenvalues of the operator L_α form a finite set. For values of λ not belonging to the spectrum, the resolvent $(L_\alpha - \lambda 1)^{-1}$ of the operator L_α is a bounded integral operator with kernel $K(x, \xi, \lambda)$, satisfying the conditions:

$$\int_0^\infty |K(x,\xi,\lambda)|^2 \, d\xi < +\infty, \qquad \int_0^\infty |K(x,\xi,\lambda)|^2 \, dx < +\infty.$$

Consider the auxiliary boundary-value problem on the interval $[0, b]$:

$$l(y) = \lambda y,$$

$$u_\nu(y) = \alpha_{\nu 0} y(0) + \alpha_{\nu 1} y'(0) + \dots + \alpha_{\nu,\, 2n-1} y^{(2n-1)}(0) = 0,$$

$$u_{\mu b}(y) = \gamma_{\mu 0} y(b) + \gamma_{\mu 1} y'(b) + \dots + \gamma_{\mu,\, 2n-1} y^{(2n-1)}(b) = 0, \tag{10}$$

$$\nu, \mu = 1, 2, \dots, n.$$

The boundary-value problem adjoint to (10) is constructed by means of the differential expression (4) and the boundary conditions adjoint to (10):

$$v_\nu(z) = \beta_{\nu 0} z(0) + \beta_{\nu 1} z'(0) + \dots + \beta_{\nu,\, 2n-1} z^{(2n-1)}(0) = 0,$$

$$v_{\mu b}(z) = \delta_{\mu 0} z(b) + \delta_{\mu 1} z'(b) + \dots + \delta_{\mu,\, 2n-1} z^{(2n-1)}(b) = 0, \tag{11}$$

$$\mu, \nu = 1,\ 2, \dots, n.$$

Denote

$$R_n = \begin{vmatrix} u_{1b}(y_{n+1}) & u_{1b}(y_{n+2}) \dots & u_{1b}(y_{2n}) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ u_{nb}(y_{n+1}) & u_{nb}(y_{n+2}) \dots & u_{nb}(y_{2n}) \end{vmatrix}, \qquad R_{n+1} = \begin{vmatrix} u_{1b}(y_n) & u_{1b}(y_{n+2}) \dots & u_{1b}(y_{2n}) \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ u_{nb}(y_n) & u_{nb}(y_{n+2}) \dots & u_{nb}(y_{2n}) \end{vmatrix}.$$

We shall assume that for $\rho \in T_k$, $R_n \neq 0$, $R_{n+1} \neq 0$. For $\rho \in \bar{S}_k$ define the single-valued holomorphic function $\omega(\rho)$ by the relations:

$$\omega(\rho) = \ln\left[\frac{R_n}{R_{n+1}} \frac{A(\rho)}{\bar{A}(\rho)}\right], \qquad \lim_{\rho\to\infty} \omega(\rho) = i \arg \frac{\theta_{-1}}{\theta_1}, \tag{12}$$

$$\theta_1 = \begin{vmatrix} \omega_1^{2n-1} \dots & \omega_{n-1}^{2n-1} & \omega_{n+1}^{2n-1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \omega_1^n \dots & \omega_{n-1}^n & \omega_{n+1}^n \end{vmatrix} \cdot \begin{vmatrix} \omega_n^{2n-1} & \omega_{n+2}^{2n-1} \dots & \omega_{2n}^{2n-1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \omega_n^n & \omega_{n+2}^n \dots & \omega_{2n}^n \end{vmatrix},$$

$$\theta_{-1} = \begin{vmatrix} \omega_1^{2n-1} \dots & \omega_n^{2n-1} \\ \cdot & \cdot \\ \cdot & \cdot \\ \omega_1^n \dots & \omega_n^n \end{vmatrix} \cdot \begin{vmatrix} \omega_{n+1}^{2n-1} \dots & \omega_{2n}^{2n-1} \\ \cdot & \cdot \\ \cdot & \cdot \\ \omega_{n+1}^n \dots & \omega_{2n}^n \end{vmatrix}.$$

For sufficiently large b, to each of the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_r$ of the operator L_α there corresponds exactly one eigenvalue $\lambda_1(b), \lambda_2(b), \dots, \lambda_r(b)$ of the boundary-value problem (10) such that $\lambda_k(b) \to \lambda_k$ as $b \to \infty$. All the remaining eigenvalues of the boundary-value problem (10) satisfy the asymptotic relation

$$\lambda = -\rho_k^{2n}, \qquad \rho_k \omega_n = \frac{k\pi i}{b} + \frac{1}{2b}\omega\left(\frac{k\pi i}{\omega_n b}\right) + \frac{1}{b} o(1) \tag{13}$$

as $b \to \infty$, uniformly with respect to ρ_k in the region

$$|\operatorname{Re}(\rho_k \omega_n)| \leq \varepsilon_1, \qquad 0 \leq |\rho_k| \leq N.$$

Let $y_k(x)$ be an eigenfunction of the operator L_α corresponding to the eigenvalue λ_k, $k = 1, 2, \dots, r$;

$$y_k(x) = -\sum_{i=1}^{n-1} \frac{\Delta_i}{\Delta_0} y_i(x,\rho) + y_n(x,\rho), \tag{14}$$

where

$$\Delta_i = \begin{vmatrix} u_1(y_1)\dots & u_1(y_{i-1}) & u_1(y_n) & u_1(y_{i+1})\dots & u_1(y_{n-1}) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ u_{n-1}(y_1)\dots & u_{n-1}(y_{i-1}) & u_{n-1}(y_n) & u_{n-1}(y_{i+1})\dots & u_{n-1}(y_{n-1}) \end{vmatrix},$$

$$\Delta_0 = \begin{vmatrix} u_1(y_1)\dots & u_1(y_{n-1}) \\ \cdot & \cdot \\ \cdot & \cdot \\ u_{n-1}(y_1)\dots & u_{n-1}(y_{n-1}) \end{vmatrix},$$

$z_k(x,\rho)$, $y_k(x,b)$, $z_k(x,b)$, $k = 1,2,\dots,r$, are the eigenfunctions of the operator L^*_α and of the boundary-value problems (10) and (11), constructed in an analogous way. Then, as $b \to \infty$, the relation

$$\frac{y_k(y,b)\,\bar{z}_k(\xi,b)}{\int_0^b y_k(x,b)\bar{z}_k(\xi,b)\,dx} = \frac{y_k(x)\bar{z}_k(\xi)}{\int_0^\infty y_k(x)\bar{z}_k(x)\,dy} + o(1), \qquad k = 1,2,\dots,r, \tag{15}$$

holds.

uniformly with respect to x, $0 \le x \le c$, $c > 0$. Denote by $y(x,\rho_k)$, $z(x,\rho_k)$, $k = 1,2,\dots$, the eigenfunctions of problems (10), (11) corresponding to the eigenvalues (13). Then, as $b \to \infty$,

$$\frac{1}{b}\int_0^b y(x,\rho_k)\,\bar{z}(x,\rho_k)\,dx = -\left[\frac{A(\rho_k)}{\widetilde{A}(\rho_k)} + \frac{B(\rho_k)}{\widetilde{B}(\rho_k)}\right] + o(1), \tag{16}$$

where $B(\rho)$, $\widetilde{B}(\rho)$ are functions constructed analogously to the functions $A(\rho)$, $\widetilde{A}(\rho)$ for L^*_α.

Let $G(x,\xi,\lambda)$ be the resolvent kernel of the boundary-value problem (10). If λ does not belong to the spectrum of the operator L_α, then, as $b \to \infty$,

$$G(x,\xi,\lambda) = K(x,\xi,\lambda) + o(1) \tag{17}$$

uniformly with respect to x,ξ in every finite square $0 \le x,\xi \le c$, $c > 0$.

Denote

$$\tilde{y}(x,\rho) = -\sum_{i=1}^{n-1} \frac{\Delta_i}{\Delta_0} y_i(x,\rho) + y_n(x,\rho) + \frac{A(\rho)}{\widetilde{A}(\rho)} \left[\sum_{i=1}^{n-1} \frac{\Delta_i'}{\Delta_0} y_i(x,\rho) - \hat{y}_{n+1}(x,\rho) \right], \tag{18}$$

where

$$\Delta_i' = \begin{vmatrix} u_1(y_1) & \cdots & u_1'(y_{i-1}) & u_1(\hat{y}_{n+1}) & u_1(y_{i+1}) & \cdots u_1(y_{n-1}) \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ u_{n-1}(y_1) & \cdots & u_{n-1}(y_{i-1}) & u_{n-1}(\hat{y}_{n+1}) & u_{n-1}(y_{i+1}) & \cdots u_{n-1}(y_{n-1}) \end{vmatrix},$$

and let $\tilde{z}(x,\rho)$ denote the function constructed analogously for the operator L_α^*. Then the following holds.

Theorem 2. Suppose that conditions (6) and (9) are satisfied, and let $K(x,\xi,\lambda)$ be the resolvent kernel of the operator L_α. For any point λ not belonging to the spectrum of the operator L_x:

$$K(x,\xi,\lambda) = \sum_{k=1}^{r} \frac{y_k(x)\overline{z_k}(\xi)}{(\lambda-\lambda_k)\int_0^\infty y_k(x)\overline{z_k}(x)\,dx} + \frac{\omega_n}{\pi i}\int_{\Gamma_k} \frac{\tilde{y}(x,\rho)\tilde{z}(\xi,\rho)\,d\rho}{(\rho^{2n}+\lambda)\left[\frac{A(\rho)}{\widetilde{A}(\rho)} + \frac{B(\rho)}{\widetilde{B}(\rho)}\right]}, \tag{19}$$

where the integral converges absolutely and uniformly with respect to x,ξ in the region $0 \le x,\xi < \infty$.

Denote by $\mathfrak{M}_\alpha$ the totality of all functions $g(x)$ satisfying the conditions: 1) $g(x)$, $l(g)$ are summable on the interval $[0,\infty)$; 2) $g^{(\nu)}(x)$, $\nu = 1,\dots,2n-1$, exist and are absolutely continuous on each finite interval $[0,a]$; 3) $u_\nu(g) = 0,\ \nu = 1,\dots,n$. Then, with the aid of formula (19) and by repeating the arguments given in (1), it is easy to obtain the expansion of the function $g(x)$ in the eigenfunctions of the operator L_α and an analogue of Parseval' s equality.

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CITED LITERATURE

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2. M. A. Naimark, *Linear Differential Operators*, 1954.

Note: Figure translations are in progress. See original paper for figures.