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Abstract

Full Text

MATHEMATICS

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ON A THEOREM OF M. RIESZ

(Presented by Academician S. L. Sobolev on 20 XI 1959)

In the study of concrete equations (for example, linear integral equations), the functional space in which the operators composing the equation act is usually not uniquely determined by the conditions of the problem. In many cases one and the same operator has to be considered in various functional spaces. In this connection, theorems are of substantial interest in which, from known properties of an operator considered in some functional spaces, one can draw conclusions about properties of the same operator considered in other spaces. A nontrivial example here is provided by the well-known theorem of M. Riesz, which is formulated below.

Let A be a linear operator acting from the space L^{p_1} into the space L^{r_1} and, at the same time, acting from the space L^{p_2} into the space L^{r_2} . Here $1 \leq p_1, r_1, p_2, r_2 \leq \infty$, and the numbers p_1, r_1, p_2, r_2 are in no way related to one another.

As usual, L^p for $1 \leq p < \infty$ is the space of functions $\varphi(x)$, summable to the p -th power on some bounded closed set G of finite-dimensional Euclidean space, with norm

$$\|\varphi\|_p = \left\{ \int_G |\varphi(x)|^p dx \right\}^{1/p}.$$

L^∞ is the space of functions essentially bounded on G , with norm

$$\|\varphi\|_\infty = \text{vrai max } |\varphi(x)|.$$

Denote by M_1 the norm of A , considered as an operator from L^{p_1} into L^{r_1} , and by M_2 the norm of A as an operator acting from L^{p_2} into L^{r_2} :

$$M_i = \sup_{\|\varphi\|_{p_i} \leq 1} \|A\varphi\|_{r_i} \quad (i = 1, 2).$$

Introduce the numbers p and r by means of the equalities

$$\frac{1}{p} = \frac{t}{p_1} + \frac{1-t}{p_2}, \quad \frac{1}{r} = \frac{t}{r_1} + \frac{1-t}{r_2}, \quad (1)$$

where t is some fixed (but arbitrary) number from the interval $(0, 1)$. It is clear that the operator A is defined on all functions from L^p , since $L^p \subset L^{p_1} \cup L^{p_2}$. The values of the operator A obviously belong to $L^{r_1} \cup L^{r_2}$. In fact, a stronger assertion holds, discovered by M. Riesz:

The operator A acts from L^p to L^r and is a bounded operator from L^p to L^r , and its norm satisfies the inequality

$$\|A\|_{L^p \rightarrow L^r} = \sup_{\|\varphi\|_p \leq 1} \|A\varphi\|_r \leq M_1^t M_2^{1-t}. \quad (2)$$

There arises the natural question whether one can draw a conclusion about the complete continuity of the operator A , considered as an operator from L^p to L^r , if it is known that it is completely continuous as an operator from L^{p_1} to L^{r_1} , or continuous from L^{p_2} to L^{r_2} . The answer to this question is given by the theorems below.

1. **Theorem 1.** *Let the number r_1 be finite. Suppose that A , as an operator acting from L^{p_1} to L^{r_1} , is completely continuous, and as an operator from L^{p_2} to L^{r_2} is continuous.*

Then the operator A acts from L^p to L^r (p and r are defined by formulas (1)) and is completely continuous.

For the proof of this theorem one constructs a sequence of bounded functions $\varphi_n(x)$ ($x \in G$) which is a basis simultaneously in all spaces L^p ($1 \leq p < \infty$). This common basis must have the property that the coefficients with respect to this basis (the biorthogonal system of functionals) are linear functionals on L^1 and do not depend on the space L^p in which the expansion with respect to the basis is considered. In the case $G = [0, 1]$, the required properties are possessed by the well-known Haar basis, composed of the Haar functions $\chi_m^{(k)}(x)$ (arranged in increasing order of m , and, for fixed m , in increasing order of k):

$$\chi_0^{(0)}(x) \equiv 1, \quad \chi_0^{(1)}(x) = \begin{cases} 1 & \text{for } 0 \leq x < \frac{1}{2}, \\ 0 & \text{for } x = \frac{1}{2}, \\ -1 & \text{for } \frac{1}{2} < x \leq 1, \end{cases}$$

and further, for $m = 1, 2, \dots$, $k = 1, 2, \dots, 2^m$,

$$\chi_m^{(k)}(x) = \begin{cases} \sqrt{2^m} & \text{for } \frac{2k-2}{2^{m+1}} \leq x < \frac{2k-1}{2^{m+1}}, \\ -\sqrt{2^m} & \text{for } \frac{2k-1}{2^{m+1}} < x \leq \frac{2k}{2^{m+1}}, \\ 0 & \text{for the remaining values of } x \end{cases}$$

(for a detailed description of the Haar basis see ⁽¹⁾, pp. 122-125).

Next, define the operators P_n of projection onto the linear span of the first n elements of the basis: to the elements

$$\varphi(x) = \sum_{i=1}^{\infty} c_i \varphi_i(x)$$

there are put in correspondence the elements

$$P_n \varphi(x) = \sum_{i=1}^n c_i \varphi_i(x).$$

The operators P_n are obviously completely continuous—they are finite-dimensional.

From the complete continuity of A as an operator from L^{p_1} to L^{r_1} it follows that the operators $P_n A$ satisfy the condition

$$\lim_{n \rightarrow \infty} \|P_n A - A\|_{L^{p_1} \rightarrow L^{r_1}} = 0.$$

If r_2 is finite, then the operators P_n in L^{r_2} , by the Banach theorem, are uniformly bounded: $\|P_n\|_{L^{r_2} \rightarrow L^{r_2}} \leq K$. Then it follows from (2) that

$$\lim_{n \rightarrow \infty} \|P_n A - A\|_{L^p \rightarrow L^r} \leq \lim_{n \rightarrow \infty} \|P_n A - A\|_{L^{p_1} \rightarrow L^{r_1}}^t (1 + K)^{1-t} \|A\|_{L^{p_2} \rightarrow L^{r_2}}^{1-t} = 0.$$

Consequently, A is completely continuous as an operator from L^p to L^r .

If $r_2 = \infty$, the proof becomes somewhat more complicated because the space L^∞ is nonseparable and cannot have a basis. In this case one must take care that the operators P_n be defined on all of L^∞ and that their norms be bounded. In the case $G = [0, 1]$, the Haar basis has the required properties.

2. For the case in which r_1 is infinite, the proof of the complete continuity of A as an operator from L^p to L^r remains valid if the system $\{\varphi_n(x)\}$ is a basis in the closure of its linear span in L^∞ (in the case $G = [0, 1]$, the Haar basis has this property) and if the values of the operator A on

L^{p_1} lie in the closure in L^∞ of the linear span of the basis. We state the corresponding assertion for the case in which the functions $A\varphi(x)$ are continuous for $\varphi(x) \in L^{p_1}$.

Theorem 2. *Let A , as a linear operator acting from the space L^{p_1} into the space C of functions continuous on G , be completely continuous, and let it be continuous as an operator from L^{p_2} into L^{r_2} .*

Then the operator A acts from the space L^p , where p is defined by the first equality (1), into the space L^r , where $r = \frac{r_2}{1-t}$, and is completely continuous.

Theorems 1 and 2 admit a generalization to the case of a set G of infinite measure.

3. Theorems 1 and 2 are directly applicable to establishing conditions for complete continuity of the linear integral operator

$$A\varphi(x) = \int_G K(x, y)\varphi(y) dy. \quad (3)$$

In particular, by this approach one can obtain some of the criteria for complete continuity indicated by L. V. Kantorovich in ⁽²⁾.

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References

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2. L. V. Kantorovich, *Uspekhi Mat. Nauk*, **11**, no. 2 (1956).

Note: Figure translations are in progress. See original paper for figures.

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