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Abstract

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ON THE PROOF OF DOUBLE SPECTRAL REPRESENTATIONS

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The aim of the present article is to obtain confirmation of the basic assumptions made by S. Mandelstam in postulating double spectral representations. These assumptions reduce to the following: the amplitude of the process is an analytic function of two complex variables—energy and momentum transfer—throughout the whole complex domain in these variables, with the exception of cuts along the real axes. By the existing methods of the general theory of dispersion relations, double spectral representations cannot be proved, since all the information available to us in the general theory makes it possible to prove only the ordinary dispersion relations. Thus, the proof of S. Mandelstam' s representations requires the use of some additional information. Usually in such a situation perturbation theory is used, since for the amplitude corresponding to a certain Feynman graph we can obtain an exact analytic expression. In the present work we investigate the contribution to the analytic structure of the scattering amplitude from the fourth order of perturbation theory. The tool of the investigation is Dyson' s integral representation for causal commutators.

Consider the scattering of two scalar bosons with masses m and μ, described by the fields $\psi(x)$ and $\varphi(x)$. As independent variables on which the amplitude will depend, choose

$$w = \frac{(k_1 + k_2)(p_1 + p_2)}{2\sqrt{(p_1 + p_2)^2}}, \qquad \Delta^2 = -\frac{(p_1 - p_2)^2}{4}, \qquad \tau = k_1^2 = k_2^2.$$

To avoid the unphysical region, τ is fixed by the condition $\tau < -\Delta^2$. The imaginary part of the scattering amplitude is written in the following form:

$$\operatorname{Im} T(\omega, \Delta^2, \tau) = \frac{1}{2}\left[M(\omega, \Delta^2, \tau) - M(-\omega, \Delta^2, \tau)\right],$$

where $M(\omega, \Delta^2, \tau)$, up to inessential factors, is defined by the expression

$$M(\omega,\Delta^2,\tau) = \int d^4x_1 d^4x_2 \exp\frac{i}{2}\{x_2[p_2-k_2]-x_1[p_1-k_1]\}$$
$$\times\sum_n \left\langle 0\left|\frac{\delta j(-\frac{1}{2}x_2)}{\delta\psi(\frac{1}{2}x_2)}\right|p_1+k_1,n\right\rangle\left\langle n,p_1+k_1\left|\frac{\delta j(-\frac{1}{2}x_1)}{\delta\psi(\frac{1}{2}x_1)}\right|0\right\rangle, \tag{1}$$

where $j(x) = i\frac{\delta S}{\delta\varphi(x)}S^+$, and S is the scattering matrix. In what follows, all integrals of interest to us will be written up to inessential factors. Using Dyson' s theorem, one can obtain an integral representation for the Fourier transform of the retarded function

$$F^{\text{ret}}(q) = \int d^4u\int_0^\infty d\lambda^2\,\frac{\Phi(u,\lambda^2)}{(q-u)^2-\lambda^2}; \tag{2}$$

it is assumed here that

$$|F^{\text{ret}}(q)| = \left|\int d^4x\,\exp[iqx]\left\langle 0\left|\frac{\delta j(-{}^1/{}_2x)}{\delta\psi({}^1/{}_2x)}\right|p_1+k_1\right\rangle\right| \leqslant \frac{A(\delta)}{|q^0|},\qquad \operatorname{Im}q^0>\delta$$

as $q^0\to\infty$.

$\varphi(u,\lambda^2)$ is an arbitrary function in some region G and is equal to zero outside it. The region G is defined by the relations

$$[{}^1/{}_2(p_1+k_1)\pm u]^2\geqslant 0;\qquad [{}^1/{}_2(p_1^0+k_1^0)\pm u^0]\geqslant 0,$$

$$\lambda\geqslant\max\left\{0,\,3\mu-\sqrt{[{}^1/{}_2(p_1+k_1)+u]^2};\quad m+\mu-\sqrt{[{}^1/{}_2(p_1+k_1)-u]^2}\right\}. \tag{3}$$

Fig. 1
Fig. 1

Fig. 2
Fig. 2

Then, taking (1) and (2) into account, we obtain

$$M(\omega,\Delta^2,\tau) = \int\frac{d^4u_1d^4u_2d\lambda_1^2d\lambda_2^2\,\Phi(u_1,\lambda_1,u_2,\lambda_2,p_1+k_1)}{\{[{}^1/{}_2(k_1-p_1)-u_1]^2-\lambda_1^2\}\{[{}^1/{}_2(k_2-p_2)-u_2]^2-\lambda_2^2\}}, \tag{4}$$

where

$$\Phi(u_1, \lambda_1, u_2, \lambda_2, p_1 + k_1) = \sum_n \varphi_n(u_1, \lambda_1, p_1 + k_1)\varphi_n^*(u_2, \lambda_2, p_1 + k_1).$$

Let us consider the Feynman graph represented in Fig. 1. The imaginary part of the scattering amplitude corresponding to the graph in Fig. 1 is

$$\operatorname{Im} T^{(4)} = \int d^4q \, \frac{\theta(q^0)\theta(p_1^0 + k_1^0 - q^0)\delta(q^2 - \mu^2)\delta[(p_1 + k_1 - q)^2 - m^2]}{[(p_1 - q)^2 - m^2][(p_2 - q)^2 - m^2]}, \quad (5)$$

where $\theta(x) = 1$ for $x^0 > 0$ and $\theta(x) = 0$ for $x^0 < 0$.

In order that $\operatorname{Im} T^{(4)}$ be representable in the form (4), it is sufficient, as is clear from (5), to define $\lambda_1, \lambda_2, u_1$, and u_2 as follows:

$$\lambda_1^2 = \lambda_2^2 = m^2, \qquad u_1 = {}^1/_2(k_1 + p_1) - q, \qquad u_2 = {}^1/_2(k_2 + p_2) - q.$$

Then the weight function is determined by the expression

$$\Phi(u_i, \lambda_i, p_1 + k_1) = \int d^4q \, \theta(q^0)\theta(p_1^0 + k_1^0 - q^0)\delta(q^2 - \mu^2)\delta[(p_1 + k_1 - q)^2 - m^2]\times$$

$$\times \, \delta(\lambda_1^2 - m^2)\delta(\lambda_2^2 - m^2)\delta[q + u_1 - {}^1/_2(k_1 + p_1)]\delta[q + u_2 - {}^1/_2(k_2 + p_2)].$$

Carrying out the integration over q and taking into account the conservation law $k_1 + p_1 = k_2 + p_2$, we obtain

$$\Phi(u_i, \lambda_i, k_1 + p_1) = \theta[{}^1/_2(k^0 + p^0) - u_1^0] \, \theta[{}^1/_2(k^0 + p^0) + u_1^0] \, \delta(u_1 - u_2)\times$$

$$\times \, \delta\{[{}^1/_2(k+p) - u]^2 - \mu^2\} \, \delta\{[{}^1/_2(k+p) + u]^2 - m^2\} \, \delta(\lambda_1^2 - m^2)\delta(\lambda_2^2 - m^2). \quad (6)$$

Let us choose the center-of-mass system, determined by the condition $p_1 + k_1 = p_2 + k_2 = 0$, and introduce new variables

$$w^2 = (p + k)^2, \qquad \Delta^2 = -{}^1/_4(p_1 - p_2)^2, \qquad \tau = k_1^2 = k_2^2.$$

It is seen from (6) that Φ satisfies (3). In addition, Φ does not depend on Δ^2 and τ. We shall consider all expressions in spherical coordinates

$$\mathbf{u}_1 = u_1\{\cos\varphi_1 \sin\beta_1;\ \sin\varphi_1 \sin\beta_1;\ \cos\beta_1\},$$

$$\mathbf{u}_2 = u_2\{\cos\varphi_2 \sin\beta_2;\ \sin\varphi_2 \sin\beta_2;\ \cos\beta_2\}, \qquad \alpha = \varphi_1 - \varphi_2, \quad \chi = \varphi_1,$$

θ is the scattering angle (see Fig. 2). In this system

$$\begin{aligned} M(w, \Delta^2, \tau) &= \\ &= \int \frac{d^4u_1\, d^4u_2\, d\lambda_1^2\, d\lambda_2^2\, \Phi(u_1, \lambda_1, u_2, \lambda_2, w)}{\left[\left(\frac{m^2-\tau}{2w} + u_1^0\right)^2 - (\mathbf{k}_1 - \mathbf{u}_1)^2 - \lambda_1^2\right]\left[\left(\frac{m^2-\tau}{2w} + u_2^0\right)^2 - (\mathbf{k}_2 - \mathbf{u}_2)^2 - \lambda_2^2\right]} \\ &= \frac{1}{K^2(\tau)} \int du_i^0\, u_i\, du_i\, d\lambda_i^2 \int_0^{2\pi} d\alpha\, d\chi \int_0^{\pi} d\beta_1\, d\beta_2\, \frac{\Phi(u_i^0, u_i, \lambda_i^2, w)}{[X_1(\tau) - \cos(\theta - \chi)][X_2(\tau) - \cos(\chi - \alpha)]}, \end{aligned} \tag{7}$$

where

$$X_i(\tau) = \frac{K^2(\tau) + u_i^2 + \lambda_i^2 - \left[\frac{m^2-\tau}{2w} + u_i^0\right]^2}{2K(\tau)u_i \sin\beta_i}, \qquad K^2(\tau) = \frac{(w^2 + m^2 - \tau)^2 - 4m^2w^2}{4w^2}$$

is the square of the momentum of the center of inertia of two particles with masses m and $\sqrt{\tau}$. Taking (6) into account, we see that $\Phi(u_i, \lambda_i, w)$ is an invariant function in the space of the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Thus, Φ does not depend on the polar angle χ, and therefore in expression (7) one may integrate over it:

$$\int_0^{2\pi} d\chi\, \frac{1}{[X_1(\tau) - \cos(\theta - \chi)][X_2(\tau) - \cos(\chi - \alpha)]} =$$

$$= \frac{2\pi\left\{X_1(\tau)/\sqrt{X_1^2(\tau) - 1} + X_2(\tau)/\sqrt{X_2^2(\tau) - 1}\right\}}{X_1(\tau)X_2(\tau) + \sqrt{X_1^2(\tau) - 1}\sqrt{X_2^2(\tau) - 1} - \cos(\theta - \alpha)}.$$

On the mass surface $\tau = \mu^2$ we introduce a new variable

$$y = X_1(\mu^2)X_2(\mu^2) - \sqrt{X_1^2(\mu^2) - 1}\sqrt{X_2^2(\mu^2) - 1}.$$

Then all the integrations in (7), except the integrations over α and y, will give a new weight function $\widetilde{\Phi}(y, w, \cos\alpha)$, i.e. (7) can be transformed to the form

$$M(w, \Delta^2) = \int_{\min_G y}^{\infty} dy \int_0^{2\pi} d\alpha \, \frac{\widetilde{\Phi}(y, w, \cos\alpha)}{y - \cos(\theta - \alpha)}. \tag{8}$$

The domain G in our particular case is determined by the relations

$$u_1 = u_2 = u, \qquad \lambda_1^2 = \lambda_2^2 = m^2,$$

$$\left(\tfrac{1}{2}w + u^0\right)^2 - u^2 = m^2, \qquad \left(\tfrac{1}{2}w - u^0\right)^2 - u^2 = \mu^2, \qquad \left(\tfrac{1}{2}w \pm u^0\right) \geqslant 0. \tag{9}$$

Since $\widetilde{\Phi}$ does not depend on Δ^2, the domain of analyticity of $M(w, \Delta^2)$ as a function of Δ^2 can be determined by investigating the zeros of the denominator of the integrand in (8). Taking (9) into account, we see that

$$X_1 = X_2 = 1 + 2\,\frac{m^2 w^2 - (m^2 - \mu^2)^2}{(w^2 + m^2 - \mu^2)^2 - 4m^2 w^2}, \qquad \text{i.e.} \quad y = 2X^2 - 1.$$

It follows from this that $y \geqslant 1$. $M(w, \Delta^2)$ will be singular where

$$y = \cos(\theta - \alpha) \quad \text{or} \quad \cos\theta = y\cos\alpha \pm i\sqrt{y^2 - 1}\,\sin\alpha. \tag{10}$$

But in our case $\alpha = 0$, i.e. the curve of singularities will be the real curve $y = \cos\theta$ (y is real for real w^2).

Let us relate the result obtained to the usual dispersion relations. As already mentioned, in the proof of the usual dispersion–

relations, in order to bypass the unphysical region, the variable $\tau = k_1^2 = k_2^2$ is introduced, fixed by the condition $\tau < -\Delta^2$. The dispersion relations proved for such τ are then analytically continued to the physical value $\tau = \mu^2$. Such a procedure will be valid if the imaginary part of the scattering amplitude is an analytic function of τ in a narrow strip near the real axis up to the value $\mu^2 + \varepsilon$. In this case, as has been shown by the methods of the general theory of dispersion relations, a restriction on momentum transfer arises, i.e., the dispersion relations turn out to be valid only for $\Delta^2 < \Delta^2_{\max}$. However, such a restriction, as is seen from Lehmann' s work [(2)], is a consequence of the fact that in the general theory one has to regard the vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ as independent, i.e., the angle α can take all possible values from 0 to 2π. In our case there are no such possibilities, since $\alpha = 0$. Then, as is not difficult to see, analytic continuation in τ to the values $\tau = \mu^2$ does not lead to a restriction on momentum transfer.

Fig. 3

Figure 1: Fig. 3

Fig. 3

Let us recall that in the dispersion relation the imaginary part of the amplitude corresponding to the graph in Fig. 1 enters into a dispersion integral specified on the segment of the real axis from the values $w^2 = (m+\mu)^2$ to $+\infty$. We have found that for real w^2 the imaginary part of the amplitude is an analytic function of $\cos\theta$ in the entire complex plane, with the exception of a cut along the real axis.

Let us compute the boundaries of the cut in momentum transfer. For simplicity let us consider the scattering of particles with equal masses $m = \mu$. In this case

$$X = 1 + \frac{2m^2}{w^2 - 4m^2}, \qquad K^2 = \frac{1}{4}\,[w^2 - 4m^2 w^2].$$

Taking into account that $\cos\theta = 1 - 2\Delta^2/K^2$, we obtain the equation of the curve of singularities of the imaginary part of the amplitude as a function of Δ^2:

$$\Delta^2 = m^2 \frac{3m^2 - w^2}{w^2 - 4m^2}, \qquad w^2 \geqslant 4m^2.$$

It follows from this that the cut in the complex Δ^2-plane begins at the point $\Delta^2 = -m^2$ and goes to $-\infty$. Then, applying to the imaginary part of the scattering amplitude entering the dispersion integral Cauchy' s theorem with the contour in the complex Δ^2-plane shown in Fig. 3, we obtain a double spectral representation.

Let us summarize. By combining Dyson' s theorem and perturbation theory it is possible quite simply, at least in the fourth order of perturbation theory, to prove the absence of complex singularities of the imaginary part of the scattering amplitude in Δ^2 for real w^2. Relating the result obtained to the usual dispersion relations, we see that the absence of complex singularities in Δ^2 is closely connected with the absence of a restriction on momentum transfer in the usual dispersion relations. The absence of complex singularities of the imaginary part of the scattering amplitude in Δ^2 for real w^2 makes it possible to prove the existence of a double spectral representation in the fourth order of perturbation theory.

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CITED LITERATURE

1. F. Dyson, Phys. Rev., **110**, 1460 (1958).

2. H. Lehmann, Nuovo Cimento, **10**, 579 (1958).

3. N. N. Bogolyubov, B. V. Medvedev, M. K. Polivanov, *Problems of the Theory of Dispersion Relations*, 1958.

4. S. Mandelstam, Phys. Rev., **112** (1958).

5. S. Mandelstam, Phys. Rev., **115**, No. 6 (1959).

Note: Figure translations are in progress. See original paper for figures.