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Abstract

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MATHEMATICS

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ESTIMATES OF TRIGONOMETRIC SUMS WITH COMPLETELY UNIFORMLY DISTRIBUTED FUNCTIONS

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A function $f(x)$ is called completely uniformly distributed ([1]) if, for every $s \geqslant 1$, for any choice of integers $m_1, \dots, m_s$ not all simultaneously equal to zero, the estimate

$$\sum_{x=1}^{P} e^{2\pi i F_s(x)} = O(P), \tag{1}$$

holds, where the function $F_s(x)$ is defined by the equality

$$F_s(x) = m_1 f(x+1) + \dots + m_s f(x+s). \tag{2}$$

In the paper ([2]), completely uniformly distributed functions are indicated for which the estimate

$$\sum_{x=1}^{P} e^{2\pi i F_s(x)} = O(P^{3/4} \ln P) \tag{3}$$

is valid.

In the present paper completely uniformly distributed functions are constructed for which the estimates of the corresponding trigonometric sums are sharper than estimate (3), and the result obtained no longer admits any further substantial improvement. The proof is based on a combination of considerations contained in the papers ([3,4]).

Let, for $k = 1, 2, \dots$, primes p_k be given from the interval $(2k+1)^k < p_k < 2(k+1)^k$. It is easy to show that there exist integers $a_{k,\nu}$ ($\nu = 1, 2, \dots, k$) such that, for $|x_\nu| \leqslant k$, the congruence

$$a_{k,1} x_1 + \dots + a_{k,k} x_k \equiv 0 \pmod{p_k}$$

has no solutions except the trivial one*. Further, let $\psi(k) > k^2$ be an arbitrary integer-valued function, $\tau_0 = 0$ and $\tau_k = \tau_{k-1} + k\psi(k)p_k$. Every integer $x \geqslant 1$ can, evidently, be represented uniquely in the form

$$x = \tau_{k-1} + ky + z,$$

where $k \geqslant 1, \quad 0 \leqslant y \leqslant \psi(k)p_k - 1$ and $1 \leqslant z \leqslant k$.

Theorem 1. The fractional parts of every function $f(x)$ defined by the equalities

$$x = \tau_{k-1} + ky + z, \qquad f(x) = \frac{a_{k,z}}{p_k}\, y,$$

are completely uniformly distributed.

* One may, for example, choose $a_{k,\nu} = (2k+1)^{\nu-1}$.

Proof. Suppose that for $\nu = 1, 2, \dots, s$ at least one of the quantities m_ν is nonzero and $|m_\nu| \le m$. Consider the sum

$$S_k = \sum_{u=1}^{rkp_k} e^{2\pi i F_s(\tau_{k-1}+u)}, \tag{4}$$

where $r \le \psi(k) - 1$ and the function F_s is defined by equality (2). We shall show that for $k \ge k_0$, where $k_0 = \max(m, s)$, the sums S_k vanish. Indeed, noting that for $0 \le y \le \psi(k)p_k - 2$ and $1 \le z \le k$, for $\nu = 1, 2, \dots, s$ the equalities

$$f(\tau_{k-1} + ky + z + \nu) = \begin{cases} \dfrac{a_{k,z+\nu}}{p_k}\, y, & \text{if } \nu \le k - z, \\ \dfrac{a_{k,z+\nu-k}}{p_k}\,(y+1), & \text{if } \nu > k - z, \end{cases}$$

hold, we obtain

$$F_s(\tau_{k-1} + ky + z) = \tag{5}$$

$$= \begin{cases} \dfrac{(m_1 a_{k,z+1} + \dots + m_{s a_{k,z+s}})y}{p_k}, & \text{if } z \le k - s, \\ \dfrac{(m_1 a_{k,z+1} + \dots + m_{k-z} a_{k,k})y + (m_{k-z+1} a_{k,1} + \dots + m_{s a_{k,z+s-k}})(y+1)}{p_k}, & \text{if } z > k - s. \end{cases}$$

Let us split the interval of summation of the sum S_k into parts:

$$|S_k| = \left|\sum_{z=1}^{k}\sum_{y=0}^{rp_k-1} e^{2\pi i F_s(\tau_{k-1}+ky+z)}\right| \leq$$

$$\leq \sum_{z=1}^{k-s}\left|\sum_{y=0}^{rp_k-1} e^{2\pi i F_s(\tau_{k-1}+ky+z)}\right| + \sum_{z=k-s+1}^{k}\left|\sum_{y=0}^{rp_k-1} e^{2\pi i F_s(\tau_{k-1}+ky+z)}\right|.$$

Using the choice of the quantities $a_{k,z}$, from (5) we obtain

$$|S_k| \leq \sum_{z=1}^{k-s}\left|\sum_{y=0}^{rp_k-1} e^{2\pi i \frac{m_1 a_{k,z+1}+\cdots+m_s a_{k,z+s}}{p_k} y}\right| +$$

$$+ \sum_{z=k-s+1}^{k}\left|\sum_{y=0}^{rp_k-1} e^{2\pi i \frac{m_1 a_{k,z+1}+\cdots+m_{k-z} a_{k,k}+m_{k-z+1} a_{k,1}+\cdots+m_s a_{k,z+s-k}}{p_k} y}\right| = 0.$$

Now estimate the sum

$$S = \sum_{x=1}^{P} e^{2\pi i F_s(x)}.$$

Define n from the condition $\tau_{n-1} < P \leq \tau_n$ and represent P in the form

$$P = \tau_{n-1} + r_1 n p_n + r_2 \qquad (0 \leq r_1 \leq \psi(n)-1, \quad 1 \leq r_2 \leq np_n).$$

Choosing in (4) $r = \psi(k)-1$ for $k < n$ and $r = r_1$ for $k = n$, we obtain

$$|S| \leq \tau_{k_0-1} + \sum_{k=k_0}^{n-1}\left|\sum_{u=1}^{\psi(k)kp_k} e^{2\pi i F_s(\tau_{k-1}+u)}\right| + \left|\sum_{u=1}^{r_1 np_n} e^{2\pi i F_s(\tau_{n-1}+u)}\right| + np_n \leq$$

$$\leq \sum_{k=k_0}^{n} |S_k| + \tau_{k_0-1} + k_0 p_{k_0} + \cdots + np_n. \tag{6}$$

Using the choice of p_k, it is easy to verify that $p_{k-1} \ll \frac{1}{k} p_k$, and, consequently,

$$k_0 p_{k_0} + \cdots + np_n \ll \frac{n(n-1)}{2} p_{n-1} + np_n \ll 2(n-1)p_n.$$

Further, in view of the fact that for $k \geq k_0$, $S_k = 0$, from (6) we obtain

$$|S| \ll \tau_{k_0-1} + 2(n-1)p_n. \tag{7}$$

On the other hand, using the fact that

$$p_n < 2(2n+1)^n = 2(2n+1)\left(1+\frac{2}{2n-1}\right)^{n-1}(2n-1)^{n-1} <$$

$$< 30(n-1)p_{n-1} \leq 30\,\frac{(n-1)\psi(n-1)p_{n-1}}{(n-1)^2} < \frac{30}{(n-1)^2}P,$$

we obtain

$$|S| < \tau_{k_0-1} + \frac{60}{n-1}P = O(P),$$

which proves the theorem.

Theorem 2. Whatever monotone function $\varphi(P)$, tending arbitrarily slowly to infinity as $P \to \infty$, there exists a completely uniformly distributed function $f(x)$ such that

$$S = \sum_{x=1}^{P} e^{2\pi i F_s(x)} = O(\varphi(P)).$$

For no completely uniformly distributed function can this estimate be improved to $O(1)$.

Proof. Denote by $\widetilde{\varphi}(k)$ the function inverse to the function $\varphi(k)$, and choose $\psi(k) = [\widetilde{\varphi}(k^2 p_{k+1})] + 1$. Obviously, defining n, as before, by the condition $\tau_{n-1} < P \leq \tau_n$, we obtain

$$\widetilde{\varphi}(n-1)^2 p_n < \psi(n-1) < P, \qquad (n-1)^2 p_n < \varphi(P).$$

Hence, in view of (7), the first assertion of the theorem follows:

$$|S| \ll \tau_{k_0-1} + \frac{2}{n-1}\varphi(P) = O(\varphi(P)).$$

Suppose that the second assertion of the theorem is false. Then there exists a completely uniformly distributed function $f(x)$ such that, for every $P \geq 1$, the estimate

$$\left|\sum_{x=1}^{P} e^{2\pi i F_s(x)}\right| < M, \tag{8}$$

holds, where M does not depend on P.

Choose $n > 2M$. Since the function $f(x)$ is completely uniformly distributed, for any natural numbers s and n the fractional parts of the system of functions $f(x+1), \dots, f(x+n+s)$ are uniformly distributed in the unit $(n+s)$-dimensional cube. Consequently, there exists x_0 such that the inequalities

$$\{f(x_0+\nu)\} < \frac{1}{2\pi(n+1)ms} \qquad (\nu = 1, 2, \dots, n+s), \tag{9}$$

hold, where $m = \max_{1\le\nu\le s} |m_\nu|$. Denote by σ_n the sum

$$\sigma_n = \sum_{x=x_0}^{x_0+n} e^{2\pi i F_s(x)}.$$

Using estimate (8), we obtain

$$|\sigma_n| = \left|\sum_{x=1}^{x_0+n} e^{2\pi i F_s(x)} - \sum_{x=1}^{x_0-1} e^{2\pi i F_s(x)}\right| < 2M. \tag{10}$$

By (9), for every x in the interval $x_0 \leqslant x \leqslant x_0+n$ it follows that

$$\left|1 - e^{2\pi i F_s(x)}\right| = 2\left|\sin\pi\big(m_1\{f(x+1)\} + \dots + m_s\{f(x+s)\}\big)\right| < \frac{1}{n+1}.$$

But then

$$|\sigma_n| = \left|n+1 - \sum_{x=x_0}^{x_0+n}\left(1 - e^{2\pi i F_s(x)}\right)\right| \geqslant$$

$$\geqslant n+1-(n+1)\max_{x_0\leqslant x\leqslant x_0+n}\left|1-e^{2\pi i F_s(x)}\right| > n.$$

Hence, by (10), the inequality $n < 2M$ follows, contradicting the choice of n, and the theorem is thereby completely proved.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.