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Abstract

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ON THE EIGENFUNCTIONS OF A CERTAIN INTEGRAL EQUATION

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I. N. Vekua (^{1}) constructed a theory of solutions of the elliptic system of differential equations

$$U_{\bar z} = |AU_* + \overline{B\bar{U}} + C_*. \tag{1}$$

An important place in the construction of this theory was occupied by the proof of the fact that the equation

$$U(z) + \frac{1}{\pi} \iint_T \frac{B(t)\overline{U(t)}}{t-z}\, dT = 0 \tag{2}$$

has no nontrivial solutions.

In constructing the theory of generalized analytic functions on Riemann surfaces, consideration of a certain integral equation analogous to (2) is of great significance. The present note contains an example of such an equation having eigenfunctions.

Let the contour Γ bound, on a closed Riemann surface R of genus p, a domain T. Let $A(\zeta, z)$ be an elementary function of the first kind of the domain T, covariant in ζ, invariant in z, analytic in both variables and having in T a single pole of the first order with residue $+1$ at the point $P[\zeta] = P[z]$. One of the possible constructions of such a function is given in (^{2}).

For what follows, the behavior of $A(\zeta, z)$ in the cylindrical domain $(R-T)_\zeta \times (R-T)_z$ is important. In this domain $A(\zeta, z)$ has a pole of the first order with residue -1 at the point $P[\zeta] = P_0$, provided that $P[z] \neq P_0$. At the point $P[\zeta] = P[z] = P_0$, the function $A(\zeta, z)$ is regular. There are poles of the first order at the points $P[z] = P_\mu$ $(\mu = 1, 2, \ldots, p)$, provided that $P[\zeta] \neq P_\mu$ $(\mu = 1, 2, \ldots, p)$.* In the case $P[z] = P[\zeta] = P_\mu$, the function $A(\zeta, z)$ is regular. There is a pole of the first order with residue $+1$ at the point $P[\zeta] = P[z] \neq P_\nu$, $(\nu = 0, 1, \ldots, p)$. In addition, we shall assume that, for $P[z_0] = P_0$, $A(\zeta, z_0) \equiv 0$. For this purpose it is enough to subtract from the function constructed in (^{2}) the covariant of the first kind $A(\zeta, z_0)$. The point P_0 is chosen arbitrarily in

$R - T$. The points P_μ ($\mu = 1, 2, \dots, p$) are also chosen with a large degree of arbitrariness, more precisely:

Theorem 1. *In order that a system of points be able to be the set of singularities (in z) of the function $A(\zeta, z)$, it is necessary and sufficient that there not exist a covariant of the first kind $Z'(z)$ vanishing at all these points.*

Proof. Necessity of the condition*. Let there exist a covariant $Z'(z)$ having zeros at the indicated points. Then the product $A(\zeta, z)Z'(z)$ is a covariant in z, having a single pole at $z = \zeta$. Hence it follows that $Z'(z) \equiv 0$.

In particular, there cannot be fewer than p such points, since for any $p-1$ points there exists a covariant possessing the indicated property (4).

Sufficiency of the condition follows from the properties of this set of points (2). The indicated points form a divisor Δ (3). Obviously, $\operatorname{ord}\Delta = p$, $\dim(W/\Delta) = 0$. Therefore, by the Riemann-Roch theorem, $\dim\Delta = 1$, i.e. on R there exists no rational function having poles only at the points of Δ.

Consider the system of differential equations in T

$$U_{\bar z} = B(z)\overline{U}, \tag{3}$$

where the coefficient depends on the local parameter as follows:

$$B^*(z^*) = B(z)\frac{d\bar z}{d\bar z^*},$$

where z and z^* are local parameters. For constructing the theory of this equation, the basic role is played by the equation

$$U(z) + \frac{1}{\pi}\iint_T B(t)\overline{U(t)}\,A(t, z)\,dT = 0. \tag{4}$$

Theorem 2. *If there exists a solution $U_0(z)$, regular in T, of equation (3), analytically continuable into $R - T$ in such a way that in this domain it is a multiple of the divisor $1/\Delta$ and has a zero at the point P_0, then equation (4) is solvable.*

Consider the function

$$H(z) \equiv U_0(z) + \frac{1}{\pi}\iint_T B(t)\overline{U_0(t)}\,A(t, z)\,dT. \tag{5}$$

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* In (2) it is proved that the number of these points does not exceed p. The fact that there are exactly p of them was proved by S. Ya. Gusman. This will, in particular, follow from Theorem 1.

Obviously, this is a function analytic on R, a multiple of the divisor $1/\Delta$. In view of the fact that $\dim \Delta = 1$, $H(z) \equiv \text{const}$. At the point P_0 the right-hand side of (5) vanishes; therefore $H(z) \equiv 0$. Consequently, $U_0(z)$ is an eigenfunction of equation (4), $\lambda = 1/\pi$. It remains to construct a function satisfying the requirements of Theorem 2.

In the domain T consider the system

$$W_{\bar{z}} = D(z)w, \tag{6}$$

where $D^*(z^*) = D(z)\, d\bar{z}/d\bar{z}^*$. We shall construct a solution W_0, regular in T, of this system, analytically continuable into $R - T$ as in Theorem 2. The function $W_0(z)$ satisfies the system

$$W_{0\bar{z}} = \tilde{B}(z)\overline{W_0}, \tag{6'}$$

where

$$\tilde{B}(z) = D(z)W_0(z)/\overline{W_0(z)}.$$

Thus, the construction of the required example is reduced to the construction of a function $W_0(z)$, for which we use the theory of the Riemann problem ([5,6]). Put $T^+ = T$ and $T^- = R - T - \Gamma$. The function $W_0(z)$ has in T^+ the representation

$$W_0(z) = \varphi^+(z) \exp \omega(z),$$

where $\varphi^+(z)$ is regular analytic in T^+, and

$$\omega(z) = -\frac{1}{\pi} \iint_T D(t)\, A(t,z)\, dT.$$

In T^- the desired function has the representation $W_0(z) = \varphi^-(z)/\Delta(z)$, where $\varphi^-(z)$ is a regular analytic function in T^- and $\Delta(z)$ is a regular analytic

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* The proof of necessity given here is due to S. Ya. Gusman.

into a T^--function whose zeros are determined by the divisor Δ. We arrive on Γ at the problem

$$\varphi^+(t) = \frac{e^{-\omega(t)}}{\Delta(t)} \varphi^-(t). \tag{7}$$

The index of this problem is equal to p; consequently, by (6), this problem is solvable.

In order to normalize the solution at the point P_0, it is sufficient to have at least two linearly independent solutions. Problem (7) has no more than $p+1$ solutions(5). Let $G(t)$ be a function of index p such that the problem $\psi^+(t) = G(t)\psi^-(t)$ has exactly $p+1$ solutions. Its existence follows from the results of work (5).

In order that problem (7) have the same number of solutions, it is sufficient(5) that

$$\int_\Gamma \ln\left[G(t)\Delta(t)e^{\omega(t)}\right]\, dZ_k(t) = 0 \qquad (k = 1, 2, \dots, p), \tag{8}$$

where dZ_k $(k = 1, 2, \dots, p)$ is a basis of differentials of the first kind on the surface R. We obtain

$$\int_\Gamma \omega(t)\, dZ_k = -\int_\Gamma \ln G(t)\Delta(t)\, dZ_k. \tag{9}$$

By a simple calculation we verify that

$$\int_\Gamma \omega(t)\, dZ_k = 2i \iint_{T^+} D(t)Z_k'(t)\, dT \qquad (k = 1, \dots, p). \tag{10}$$

Choosing $D(z)$ so that condition (8) is satisfied, we obtain that problem (7) has $p+1$ solutions, which makes it possible, for $p > 1$, to obtain the required normalization.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.