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Abstract

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ON HOMOLOGY AND COHOMOLOGY GROUPS OVER A PAIR OF COEFFICIENT GROUPS

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The homological, as well as the cohomological, theory of an infinite complex, as is known, is constructed on the group of all finite or all infinite chains, which gives two different theories of homology and, respectively, cohomology of a complex. This also occurs for groups of a space, in the definition of which infinite complexes are used, for example nerves of infinite coverings, etc. The definitions of the homology and cohomology groups of a complex and of a space given below are based on the consideration of certain families of chains taken with respect to a pair of coefficient groups. As a result we obtain a whole series of groups, in which the groups of the two theories mentioned above occupy the extreme positions. Under a certain relation between pairs of coefficient groups there is a duality between homology and cohomology groups, generalizing the known dualities.

1. Let a locally finite complex K and a pair of discrete or compact groups (X, X') be given, of which X' is a subgroup of the group X. By a p-dimensional chain of the complex K with respect to the pair of coefficient groups (X, X') we mean a function c_p, sometimes written in the form $\sum_i x^i t_i^p$, which assigns to each p-dimensional simplex t_i^p of K a definite element x^i of the group X, it being assumed that for almost all simplexes t_i^p the coefficients x^i belong to the subgroup X'. Defining addition of chains by the formula $c_p + c'_p = \sum_i (x^i + {x'}^i) t_i^p$, we find that the set of all such chains forms a group, which we denote by $C_p(K; X, X')$.

The group of cochains $C^p(K; Y, Y')$ of the complex K is defined in exactly the same way as the group of chains $C_p(K; X, X')$, but below it is considered with respect to the pair of coefficient groups (Y, Y'), conjugate to the pair (X, X'); here the pairs of groups (X, X') and (Y, Y') are called **conjugate** if the groups X and Y are dual to each other, $X \mid Y$, and the subgroups X' and Y' are annihilators of one another, i.e. X' is the annihilator of the subgroup Y' in X, and Y' is the annihilator of the subgroup X' in Y. Conjugacy between pairs of coefficient groups will enable us to introduce a topology in the groups $C_p(K; X, X')$ and $C^p(K; Y, Y')$, and in such a way that, for compact coefficient groups, we shall have topologizations allowing the natural compact completion of the groups under consideration; moreover, the topologized groups of chains and cochains will be dual for conjugate pairs of coefficient groups.

For this purpose let us multiply the groups $C_p(K; X, X')$ and $C^p(K; Y, Y')$, still considered without topology, regarding the product (c_p, c^p) of an element $c_p = \sum_i x^i t_i^p$ of the group $C_p(K; X, X')$ by an element $c^p = \sum_i y^i t_i^p$ of the group

$C^p(K; Y, Y')$ the sum $\sum_i x^i y^i$; this sum is meaningful, since from the conjugacy of the pairs (X, X') and (Y, Y') and from the definition of chains and cochains with respect to a pair of groups it follows that, for any c_p and c^p, for almost all i the equality $x^i y^i = 0$ holds. By assumption one of the pairs (X, X') or (Y, Y') consists of discrete groups. We introduce the discrete topology also into that one of the groups $C_p(K; X, X')$, $C^p(K; Y, Y')$ for which the pair of coefficient groups is a pair of discrete groups. Then the other group is topologized, by means of the pairing defined above, in the following way. Suppose, for example, that the pair (Y, Y') is discrete and, consequently, the group of cochains $C^p(K; Y, Y')$ is discrete. Take in this group a finite number of arbitrary cochains $c_1^p, c_2^p, \dots, c_n^p$ and some neighborhood W of zero in the group of real numbers modulo 1. By a neighborhood of zero in the group $C_p(K; X, X')$ we shall mean the set U of those chains c_p for which $(c_i^p, c_p) \in W$, $i = 1, 2, \dots, n$. Choosing in all possible ways finite subsystems $(c_1^p, c_2^p, \dots, c_n^p)$ and neighborhoods W, we obtain a system of neighborhoods of the zero chain which, as can be verified, turns $C_p(K; X, X')$ into a topological group $\widetilde{C}_p(K; X, X')$. In this topology the pairing (c_p, c^p) introduced above is continuous; and since it is, moreover, distributive and orthogonal, we obtain a topologically monomorphic natural mapping of the group $\widetilde{C}_p(K; X, X')$ into the group of characters of the discrete group $C^p(K; Y, Y')$. Hence we conclude that the group $\widetilde{C}_p(K; X, X')$ has a compact completion. By the group of chains of the complex K with respect to the compact pair of coefficient groups (X, X') we shall henceforth mean the group obtained as a result of the compact completion of the group $\widetilde{C}_p(K; X, X')$; we shall denote it by the same symbol $C_p(K; X, X')$, since in each case it will be clear whether we are dealing with the compact group of chains or with the original nontopologized group; accordingly, the elements of the group $C_p(K; X, X')$ will be called chains of the complex K with respect to the pair of groups (X, X').

It is proved that the group $C_p(K; X, X')$ thus obtained is the group of characters of the group $C^p(K; Y, Y')$, and conversely. In exactly the same way, by topologizing, in the manner described above, the original group of cochains $C^p(K; Y, Y')$, for compact groups (Y, Y') we obtain the group $\widetilde{C}^p(K; Y, Y')$, whose completion gives the compact group of cochains $C^p(K; Y, Y')$ over the pair (Y, Y'). Here we have the duality of the compact group of cochains $C^p(K; Y, Y')$ with the discrete group of chains $C_p(K; X, X')$.

Thus, the groups $C_p(K; X, X')$ and $C^p(K; Y, Y')$ are constructed in such a way that each of them is compact when it is taken over a pair of compact coefficient groups, and discrete when it is taken over a pair of discrete coefficient groups, while at the same time it has been proved that for conjugate pairs these groups are dual to one another:

$$C_p(K; X, X') \mid C^p(K; Y, Y').$$

Now in the groups $C_p(K; X, X')$ and $C^p(K; Y, Y')$ we introduce the boundary and, respectively, coboundary operators Δ and ∇. These operators are defined by means of the known formulas (see, for example, (^{1}, ^{4})); but in the case of discrete coefficient groups this is done, as usual, in the whole group of chains or cochains, whereas in the case of compact coefficient groups the operators are first introduced in the groups before completion, $\widetilde{C}_p(K; X, X')$ and $\widetilde{C}^p(K; Y, Y')$. It is then shown that any chain or cochain almost all of whose coefficients belong to the subgroup X' and, respectively, Y', is carried by the operators Δ and, respectively, ∇ into a chain and a cochain with the same property. In the case of compact

coefficient groups, the operators Δ and ∇ thereafter extend continuously from $\widetilde{C}_p(K; X, X')$ and, respectively, $\widetilde{C}^p(K; Y, Y')$ to $C_p(K; X, X')$ and $C^p(K; Y, Y')$. Thus we obtain a chain complex $\{C_p(K; X, X'), \Delta\}$ and a cochain complex $\{C^p(K; Y, Y'), \nabla\}$, each of which may consist either of discrete or of compact groups; the homomorphisms of these complexes will be the homomorphisms continuous in the introduced topologies

$$\Delta : C_p(K; X, X') \to C_{p-1}(K; X, X')$$

and

$$\nabla : C^p(K; Y, Y') \to C^{p+1}(K; Y, Y').$$

By constructing over these complexes the homological and, respectively, the cohomological functors, we obtain the homology group $H_p(K; X, X')$ and the cohomology group $H^p(K; Y, Y')$ of the infinite complex K over a pair of coefficient groups. Each of these groups will be discrete or compact, according as the coefficient groups with respect to which they are taken are discrete or compact. On the basis of the duality of the groups $C_p(K; X, X')$ and $C^p(K; Y, Y')$ proved above, and of the equality

$$(\Delta c_p, c^{p-1}) = (c_p, \nabla c^{p-1}),$$

where $c_p \in C_p(K; X, X')$, $c^{p-1} \in C^{p-1}(K; Y, Y')$, we shall show that, in the case when the pairs of coefficient groups are conjugate, the introduced homology and cohomology groups are dual to one another:

$$H_p(K; X, X') \mid H^p(K; Y, Y').$$

Let us consider two particular limiting cases which arise here. If $X' = X$ and, consequently, $Y' = 0$, then $C_p(K; X, X)$ is the group of infinite chains

$C_p(K;X)$, while $C^p(K;Y,0)$ is the group of finite cochains $C^p(K;Y)$; therefore $H_p(K;X,X)$ is the homology group of infinite cycles, and $H^p(K;Y,0)$ is the cohomology group of finite cocycles of the complex K. If $X' = 0$, then $Y' = Y$, whence $C_p(K;X,0)$ is the group of finite chains over X, and $C^p(K;Y,Y)$ is the group of infinite cochains over Y; consequently, $H_p(K;X,0)$ is the homology group of finite cycles, and $H^p(K;Y,Y)$ is the cohomology group of infinite cocycles. In both cases the groups based on finite chains or cochains and taken over a compact coefficient group have a compact topology, which is introduced into these groups by means of passage to the limit in the sense of Chogoshvili in direct spectra of compact groups ([3],[7]).

2. For the construction of the groups of a space relative to a pair of coefficient groups, we take the collection $\{O_\alpha\}$ of all open covers O_α of the topological space R, and for each cover O_α its Vietoris complex K_α; this is a simplicial complex whose simplices have as vertices points of the space R, and a given finite set of vertices forms a simplex if it is contained in one and the same element of the cover O_α. The Vietoris complex is not a star-finite complex, and therefore the usual definitions do not give homology groups for it (see ([5],[6])); for the same reason, over a pair of coefficient groups there cannot be defined for the Vietoris complex not only homological, but also cohomological groups, since the coboundary of a cochain almost all of whose coefficients belong to Y' may have an infinite number of coefficients from $Y \setminus Y'$. In view of this we must give another definition of the groups of a non-star-finite complex.

On each Vietoris complex K_α consider, ordered by increasing refinement, the system $\{K_{\alpha a}\}$ of all its star-finite subcomplexes $K_{\alpha a}$, i.e. we shall assume that $a < b$ if $K_{\alpha a} \subset K_{\alpha b}$. This inclusion gives rise to an inclusion homomorphism

$$\pi^\alpha_{ab}$$

of the homology group $H_p(K_{\alpha a};X,X')$ of the complex $K_{\alpha a}$ relative to the pair (X,X') into the group $H_p(K_{\alpha b};X,X')$ of the complex $K_{\alpha b}$, and to a cutting homomorphism

$$\rho^\alpha_{ba}$$

of the cohomology group $H^p(K_{\alpha b};Y,Y')$ of the complex $K_{\alpha b}$ relative to the pair (Y,Y') into the group $H^p(K_{\alpha a};Y,Y')$ of the complex $K_{\alpha a}$. These groups exist, for $K_{\alpha a}$ are locally finite complexes. The indicated—

homomorphisms, together with the corresponding groups, form the inverse spectrum
$\{H^p(K_{aa};Y,Y'),\rho^\alpha_{ba}\}$ and the direct spectrum
$\{H_p(K_{aa};X,X'),\pi^\alpha_{ab}\}$. We take the limit groups of these spectra to be the groups of the complex K_α and denote them by $H^p(K_\alpha;Y,Y')$ and $H_p(K_\alpha;X,X')$; here and in what follows the limit group for an inverse spectrum of compact groups and for a direct spectrum of discrete groups is taken in the usual, classical sense (see, for example, ([1],[5],[6])), while the limit

groups of an inverse spectrum of discrete groups and of a direct spectrum of compact groups are taken in the sense of Čogoshvili (see ([2,7])). This ensures that all the limit groups obtained are compact when the coefficient groups are compact, and discrete when the coefficient groups are discrete.

If $\alpha < \beta$, i.e. O_β is inscribed in O_α, then the Vietoris complex K_β is a subcomplex of the Vietoris complex K_α. It can be shown that this inclusion induces natural homomorphisms of the groups introduced above: the inclusion homomorphism $\sigma_{\beta\alpha}$ of the group $H_p(K_\beta; X, X')$ into the group $H_p(K_\alpha; X, X')$, and the excision homomorphism $\tau_{\alpha\beta}$ of the group $H^p(K_\alpha; Y, Y')$ into the group $H^p(K_\beta; Y, Y')$; these groups and homomorphisms generate spectra: the inverse spectrum $\{H_p(K_\alpha; X, X'), \sigma_{\beta\alpha}\}$ and the direct spectrum $\{H^p(K_\alpha; Y, Y'), \tau_{\alpha\beta}\}$. The limit groups of these spectra, taken in the sense indicated above, are, by definition, the homology and, respectively, the cohomology group of the space R relative to the pair of coefficient groups (X, X') and (Y, Y'), and are denoted by $H_p(R; X, X')$ and $H^p(R; Y, Y')$. We note that in order to obtain each of these groups it was necessary to perform a passage to the limit both in the inverse and in the direct spectrum, and in each case the coefficient groups could be either discrete or compact groups; accordingly, the groups $H_p(R; X, X')$ and $H^p(R; Y, Y')$ will be discrete or compact together with the coefficient groups. We further note that even in the special cases $X' = 0$ $(Y' = Y)$ and $X' = X$ $(Y' = 0)$, the groups obtained are new invariants of the space.

It is proved that if the pairs of coefficient groups (X, X') and (Y, Y') are conjugate groups, then, independently of which of these pairs consists of compact groups and which of discrete ones, we have the duality

$$H_p(R; X, X') \mid H^p(R; Y, Y').$$

For the construction of the spectral groups of Aleksandrov-Čech of the space R over a pair of coefficient groups, we must take such a system $\{O_\alpha\}$ of star-finite open coverings O_α of the space R that, if O_β is inscribed in O_α, then every element of O_α contains no more than a finite number of elements of O_β (such will be, for example, aggregates in the sense of ([8])). On the nerves of the coverings O_α we take groups over a pair of coefficient groups and then proceed as usual.

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Note: Figure translations are in progress. See original paper for figures.