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Abstract

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ASYMPTOTICS OF ZONAL SPHERICAL FUNCTIONS ON THE SIEGEL UPPER HALF-PLANE

(Presented by Academician I. G. Petrovsky, 4.VII 1960)

1. Let G be a connected noncompact semisimple Lie group; K its maximal compact subgroup; $M = G/K$ the corresponding symmetric space; $G = KA_+K$, where A_+ is the Cartan subgroup of the space M. Let R^n ($n = \operatorname{rank} M$) be the Cartan algebra; W the Weyl group for the space M; w the order of this finite group. Finally, let P_+ be the system of all positive roots that do not vanish on $[R^n] = R^n + \sqrt{-1}\,R^n$. Put

$$\rho = \frac{1}{2} \sum_{\alpha \in P_+} \alpha.$$

Denote by φ_λ the zonal spherical function of positive-definite type corresponding to the vector $\lambda \in R^n$ ([1–3]). For every function f constant on the double cosets of the group G with respect to the subgroup K, put

$$\tilde{f}(\lambda) = \int_G f(x)\, \overline{\varphi_\lambda(x)}\, dx.$$

It is known that

$$\int_G |f(x)|^2\, dx = \frac{1}{w} \int |\tilde{f}(\lambda)|^2\, d\mu(\lambda);$$

the measure $d\mu(\lambda)$ is the (uniquely determined) Plancherel measure for the space M. Let $d\lambda$ be Euclidean measure in R^n; as shown in Harish-Chandra' s paper ([7]),

$$d\mu = \frac{1}{|c(\lambda)|^2}\, d\lambda.$$

The function $c(\lambda)$ is connected with the asymptotic behavior of the zonal spherical function φ_λ in the following way (see ([7,5])). Let $t = (t_1, \dots, t_n)$ be the canonical coordinates of an element $h \in A_+$. We shall say that $h \to \infty$ on the Cartan group A_+ if $(\alpha, t) \to +\infty$ for every $\alpha \in P_+$. Then

$$\varphi_\lambda(h) \underset{h\to\infty}{\sim} e^{-(\rho,t)} \sum_{s\in W} c(s\lambda) e^{i(\lambda, st)},$$

$$\lambda \in R^n + \sqrt{-1}R^n, \quad |\operatorname{Im}\lambda| > \varepsilon, \quad a(\lambda) \neq 0 \quad \text{for } \alpha \in P_+,$$

and moreover

$$|c(s\lambda)| = |c(\lambda)| \qquad (s \in W).$$

In the present paper the Plancherel measure is computed when G is the real symplectic group.

2. Denote by $\mathfrak{S}_n$ the space of all complex matrices $Z = X + iY$ with positive-definite imaginary part: $Y = \operatorname{Im} Z > 0$.

This domain is called the Siegel upper half-plane and is a classical domain of type (CI) in Cartan' s classification. The motion group of this domain is isomorphic to the real symplectic group $\mathrm{Sp}(n; \mathbb{R})$.

First we shall indicate the subgroups $\mathcal{K}, \mathcal{A}_+, \mathcal{N}^+, \mathcal{N}^-$ of the group $\mathrm{Sp}(n; \mathbb{R})$ that we need—subgroups typical for arbitrary semisimple Lie groups. Let E_n be the identity matrix of order n. Then, by definition,

$$\mathrm{Sp}(n;\mathbb{R}) = \{g \in GL(2n;\mathbb{R}) : gJg' = J\}, \quad \text{where} \quad J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

The subgroup $\mathcal{K}$ consists of all symplectic orthogonal matrices. The subgroup $\mathcal{A}_+$ consists of all symplectic diagonal matrices with positive entries:

$$\mathcal{A}_+ = \left\{ h = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \quad \varepsilon = \operatorname{Diag}(\varepsilon_1, \dots, \varepsilon_n), \quad \varepsilon_1 > 0, \dots, \varepsilon_n > 0 \right\}.$$

The subgroup $\mathcal{N}^+$

$$\mathcal{N}^+ = \left\{ n^+ = \begin{pmatrix} Y'^{-1} & BY \\ 0 & Y \end{pmatrix}, \quad B = B', \quad Y = \begin{pmatrix} 1 & 0 \dots 0 & \\ y_{21} & 1 \dots 0 & \\ \vdots & & \\ y_{n1} & y_{n2} \dots 1 & \end{pmatrix} \right\}.$$

The subgroup $\mathcal{N}^-$:

$$\mathcal{N}^- = \left\{ n^- = \begin{pmatrix} X & 0 \\ SX & X'^{-1} \end{pmatrix}, \quad S = S' = (\sigma_{ij}), \quad X = \begin{pmatrix} 1 & 0 \dots 0 \\ x_{21} & 1 \dots 0 \\ \vdots & \\ x_{n1} & x_{n2} \dots 1 \end{pmatrix} \right\}.$$

Then $\mathrm{Sp}(n; \mathbb{R}) = \mathcal{K}\mathcal{A}_+\mathcal{N}^+$, i.e. every element $g \in \mathrm{Sp}(n; \mathbb{R})$ is uniquely representable in the form $g = khn^+$, where $k \in \mathcal{K},\ h \in \mathcal{A}_+,\ n^+ \in \mathcal{N}^+$.

We shall call the eigenvalues of the matrix h the composite radius of the element g. Let us find the composite radius (?) $(r_1, \dots, r_n)$ of an element $n^- \in \mathcal{N}^-$; for this purpose denote by $\xi_1, \dots, \xi_n$ the columns of the matrix X, and put

$$\widetilde{D}_p = \det \|(\xi_i, T\xi_j)\|_1^p \quad (p = 1, 2, \dots, n-1),$$

$$\Delta = \det T, \qquad T = E_n + S^2.$$

Then, as is easy to see,

$$r_1^2 = \widetilde{D}_1, \qquad r_1^2 r_2^2 = \widetilde{D}_2, \dots, \qquad r_1^2 \cdots r_{n-1}^2 = \widetilde{D}_{n-1}, \qquad r_1^2 \cdots r_n^2 = \Delta.$$

Taking into account that the Cartan algebra R^n consists of matrices of the form

$$H = \mathrm{Diag}(h_1, \dots, h_n, -h_1, \dots, -h_n)$$

and

$$\rho = \frac{1}{2} \sum_{\alpha \in P_+} \alpha = ne_1 + (n-1)e_1 + \dots + 2e_{n-1} + e_n; \quad e_i(H) = h_i \ (i = 1, \dots, n),$$

we obtain

$$r_1^{i\lambda_1 + n} r_2^{i\lambda_2 + n-1} \cdots r_{n-1}^{i\lambda_{n-1}+2} r_n^{i\lambda_n + 1} = \widetilde{D}_1^{\,i\frac{\lambda_1 - \lambda_2}{2} + \frac{1}{2}} \cdots \widetilde{D}_{n-1}^{\,i\frac{\lambda_{n-1} - \lambda_n}{2} + \frac{1}{2}} \cdot \Delta^{\,i\frac{\lambda_n}{2} + \frac{1}{2}}.$$

The composite radius of the element hn^-h^{-1} is immediately obtained from the composite radius of the element n^-, when in the determinants $\widetilde{D}_p, \Delta$ we replace all x_{ij} by

$$x_{ij} \frac{\varepsilon_i}{\varepsilon_j}$$

and all σ_{ij} by

$$\frac{\sigma_{ij}}{\varepsilon_i \varepsilon_j};$$

the determinants thus obtained will be denoted by $\widetilde{D}_p(\varepsilon)$ and $\Delta(\varepsilon)$.

Theorem 1. The zonal spherical functions on the Siegel upper half-plane are given by the integrals

$$\varphi_\lambda(\varepsilon) = \frac{l}{\varepsilon_1^n \varepsilon_2^{n-1} \dots \varepsilon_{n-1}^2 \varepsilon_n} \times$$

$$\times \iint_{(X,S)} \frac{\widetilde{D}_1(\varepsilon)^{i\frac{\lambda_1-\lambda_2}{2}-\frac{1}{2}} \dots \widetilde{D}_{n-1}(\varepsilon)^{i\frac{\lambda_{n-1}-\lambda_n}{2}-\frac{1}{2}} \cdot \Delta(\varepsilon)^{i\frac{\lambda_n}{2}-\frac{1}{2}}}{\widetilde{D}_1^{i\frac{\lambda_1-\lambda_2}{2}+\frac{1}{2}} \dots \widetilde{D}_{n-1}^{i\frac{\lambda_{n-1}-\lambda_n}{2}+\frac{1}{2}} \cdot \Delta^{i\frac{\lambda_n}{2}+\frac{1}{2}}} \, dX \, dS, \qquad (1)$$

where

$$dX = \prod_{i>j} dx_{ij}, \qquad dS = \prod_{i \le j} d\sigma_{ij}.$$

Moreover, if λ is a real vector, i.e. if $\lambda \in R^n$, then φ_λ is also positive definite. The constant l is determined by the condition $\varphi_\lambda(1) = 1$.

Applying Harish-Chandra' s results ([7]), we conclude

$$\varphi_\lambda(\varepsilon) \cong \frac{1}{\varepsilon_1^n \varepsilon_2^{n-1} \dots \varepsilon_{n-1}^2 \varepsilon_n} \sum_{(k_1,\dots,k_n)} c(\pm\lambda_{k_1}, \dots, \pm\lambda_{k_n}) \varepsilon_1^{\pm i\lambda_{k_1}} \dots \varepsilon_n^{\pm i\lambda_{k_n}},$$

where the sum runs over $n!$ permutations $(k_1, \dots, k_n)$ of $(1, 2, \dots, n)$. Further,

$$|c(\pm\lambda_{k_1}, \dots, \pm\lambda_{k_n})| = |c(\lambda_1, \dots, \lambda_n)|, \qquad \operatorname{Re}\lambda_1 > \dots > \operatorname{Re}\lambda_n > 0,$$

and, if $\lambda \in R^n$, $\lambda_1 > \dots > \lambda_n > 0$,

$$c(\lambda) = c(\lambda_1, \dots, \lambda_n) = \lim_{\theta \to +0} c(\lambda_1 - in\theta, \lambda_2 - i(n-1)\theta, \dots, \lambda_n - i\theta).$$

We introduce for consideration the integral

$$C(\alpha) = C(\alpha_1, \dots, \alpha_n) = \iint_{(X,S)} \frac{dX \, dS}{\widetilde{D}_1^{\alpha_1-\alpha_2+\frac{1}{2}} \dots \widetilde{D}_{n-1}^{\alpha_{n-1}-\alpha_n+\frac{1}{2}} \cdot \Delta^{\alpha_n+\frac{1}{2}}}. \qquad (2)$$

As follows from the paper ([7]),

$$c(\lambda) = \lim_{\theta \to +0} C\left(\frac{i\lambda_1 + n\theta}{2}, \frac{i\lambda_2 + (n-1)\theta}{2}, \dots, \frac{i\lambda_n + \theta}{2}\right). \tag{3}$$

We note in passing that the constant l in formula (1) is determined by the relation

$$\frac{1}{l} = C\left(\frac{n}{2}, \frac{n-1}{2}, \dots, \frac{1}{2}\right). \tag{4}$$

Theorem 2.

$$C(\alpha) = a(\alpha)b(\alpha),$$

where

$$a(\alpha) = \int_{(X)} \frac{dX}{D_1^{\alpha_1 - \alpha_2 + \frac{1}{2}} \dots D_{n-1}^{\alpha_{n-1} - \alpha_n + \frac{1}{2}}},$$

$$b(\alpha) = \int_{(S)} \frac{\Delta_1^{\alpha_1 - \alpha_2 - \frac{1}{2}} \dots \Delta_{n-1}^{\alpha_{n-1} - \alpha_n - \frac{1}{2}}}{\Delta^{\alpha_1 + \frac{1}{2}}} \, dS,$$

$$D_p = \det \|(\xi_i, \xi_j)\|_1^p, \qquad (p = 1, \dots, n-1).$$

Here

$$\Delta = \det T, \qquad T = E_n + S^2 = (t_{ij}), \qquad \Delta_p = \det \|t_{ij}\|_{p+1}^n \quad (p = 1, \dots, n-1).$$

The integral for $a(\alpha)$ has already been computed by us in (8):

$$a(\alpha) = \prod_{1 \le i < j \le n} \mathrm{B}\left(\alpha_i - \alpha_j; \frac{1}{2}\right); \tag{5}$$

where $\mathrm{B}(x, y)$ is Euler' s beta function. The computation of the second integral will be carried out inductively.

More explicitly, put

$$S = \begin{pmatrix} \sigma & v \\ v' & \widetilde{S} \end{pmatrix}, \qquad E_{n-1} + \widetilde{S}^2 = \widetilde{T} = \|\tilde{t}_{ij}\|_2^n$$

($\widetilde{S}$ is a matrix of order $n-1$),

$$\widetilde{\Delta} = \det \widetilde{T}, \qquad \widetilde{\Delta}_p = \det \|\tilde{t}_{ij}\|_{p+1}^{n} \quad (p = 2, \dots, n-1),$$

and let

$$\tilde{b}(\alpha_2, \dots, \alpha_n) = \int_{(\widetilde{S})} \frac{\widetilde{\Delta}_2^{\alpha_2-\alpha_3-\frac{1}{2}} \dots \widetilde{\Delta}_{n-1}^{\alpha_{n-1}-\alpha_n-\frac{1}{2}}}{\widetilde{\Delta}^{\alpha_2+\frac{1}{2}}} \, d\widetilde{S}.$$

Theorem 3.

$$b(\alpha) = \tilde{b}(\alpha_2, \dots, \alpha_n)\, \mathrm{B}\left(\alpha_1; \frac{1}{2}\right) \mathrm{B}\left(\alpha_1 + \alpha_2; \frac{1}{2}\right) \cdots \mathrm{B}\left(\alpha_1 + \alpha_n; \frac{1}{2}\right).$$

It should be noted that analogous, but somewhat different, integrals, in which the argument ranges over various sets of matrices, were considered by Hua Loo-keng (see, for example, (6)).

From Theorems 2 and 3 our main assertion finally follows:

Theorem 4. *The Plancherel measure $d\mu$ in the case of spherical functions on the Siegel half-plane is given by the formula*

$$d\mu = \frac{1}{\pi^n l^2} \prod_{1 \le p < q \le n} \frac{\lambda_p - \lambda_q}{2} \operatorname{th} \frac{\lambda_p - \lambda_q}{2} \pi \cdot \prod_{1 \le p < q \le n} \frac{\lambda_p + \lambda_q}{2} \operatorname{th} \frac{\lambda_p + \lambda_q}{2} \pi \times$$

$$\times \prod_{p=1}^{n} \frac{\lambda_p}{2} \operatorname{th} \frac{\lambda_p}{2} \pi \cdot d\lambda_1 \cdots d\lambda_n,$$

where, according to (4),

$$\frac{1}{l} \prod_{1 \le p < q \le n} \mathrm{B}\left(\frac{q-p}{2}; \frac{1}{2}\right) \cdot \prod_{1 \le p < q \le n} \mathrm{B}\left(n + 1 - \frac{p+q}{2}; \frac{1}{2}\right) \times$$

$$\times \prod_{p=1}^{n} \mathrm{B}\left(\frac{n-p+1}{2}; \frac{1}{2}\right).$$

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Note: Figure translations are in progress. See original paper for figures.