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Abstract

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ON MULTIVALUED LINEAR OPERATORS IN A LOCALLY CONVEX SPACE

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In the present note we consider some properties of general, generally speaking multivalued, linear operators in locally convex separated spaces.

Let X, Y be two locally convex spaces. Any subset of the product (X, Y) of the spaces X and Y may be regarded as the graph Γ_A of some operator A with domain $D_A \subset X$ and range $R_A \subset Y$. We shall call an operator A **linear** if its graph Γ_A is a linear set. We shall call a linear operator A **open** if it maps every neighborhood of zero in D_A into a neighborhood of zero in R_A. An operator inverse to an open one will be called **continuous**. We define the **closure** $\overline{A}$ of an operator A as the operator whose graph is the closure of the graph of the given operator.

If M is a vector subspace in X, and X^* is the space conjugate to X, then by M^0 we shall denote the vector subspace in X^* orthogonal to M. By the **operator adjoint to the given linear operator** A we shall mean the operator A^* with domain $D_{A^*} \subset Y^*$ and graph

$$\Gamma_{A^*} = \Gamma^0_{-A} \subset (X^*, Y^*).$$

It is easy to verify that

$$\Gamma_{A^*} = \{(x^*, y^*) \in (X^*, Y^*) : \langle Ax, y^* \rangle = \langle x, x^* \rangle,\ x \in D_A\}.$$

Theorem 1. 1°. In order that the operator A be open, it is necessary, and if $A^{-1}(0) = \overline{A}^{-1}(0)$, also sufficient, that the operator $\overline{A}$ be open.

2°. If the set R_A is closed, then $R_A = R_{\overline{A}}$.

3°. If the operator A is open and the set R_A is closed, then

$$A^{-1}(0) = \overline{A}^{-1}(0).$$

4°. Let the operator A be open and the set R_A closed. Then, in order that the operator A be closed, it is necessary and sufficient that the set $A^{-1}(0)$ be closed.

5°. In order that the operator A be closed, it is necessary and sufficient that

$$R_A = R_{\overline{A}} \quad \text{and} \quad A^{-1}(0) = \overline{A}^{-1}(0).$$

Theorem 2.

$$A^*(0) = D_A^0, \qquad A^{*-1}(0) = R_A^0,$$

$$\overline{A}(0) = D_{A^*}^0, \qquad \overline{A}^{-1}(0) = R_{A^*}^0.$$

We shall prove the first relation. Obviously,

$$(D_A, Y) = (0, Y) + \Gamma_A.$$

Hence

$$(D_A^o, 0) = (X^*, 0) \cap \Gamma_A^o = (A^*(0), 0),$$

and the required equality is proved.

Corollaries. 1°. In order that the operator A admit a single-valued closure, it is necessary and sufficient that D_{A^*} be weakly dense in Y^*.

2°. In order that the operator A^* be single-valued, it is necessary and sufficient that D_A be dense in X.

Theorem 3. Let A be a linear closed operator and $\Gamma_A \subset (X, Y)$.

I. In order that the operator A be weakly open, it is necessary and sufficient that R_{A^*} be weakly closed.

II. Suppose that X, Y are Fréchet spaces. Then the following five properties are equivalent:

1) A is strongly open;

2) A is weakly open;

3) R_A is closed;

4) A^* is weakly open;

5) R_{A^*} is weakly closed.

III. Let X and Y be Banach spaces. Then each of the five properties in item II is equivalent to either of the following two:

6) A^* is strongly open;

7) R_{A^*} is strongly closed.

The proofs of Theorems 1 and 3 systematically use the results of the monograph ([1]).

Theorem 3 is proved by reduction to the case of a linear single-valued continuous operator defined on the whole space. For this purpose one considers the linear single-valued continuous operator V, defined on X as follows:

$$V(x) = \varphi(x, 0),$$

where φ is the canonical mapping of the space (X, Y) onto the quotient space $(X, Y)/\Gamma_A$. Next, the connections between the properties of the operators A, A^* and V, V^* are investigated. The theorem follows from the corresponding assertions for the operator V.

Remark 1. Applying Theorems 1 and 3 to the operator A^{-1} is equivalent to replacing in these theorems the sets $A^{-1}(0), \overline{A}^{-1}(0), R_A, R_{\overline{A}}, R_{A^*}$, respectively, by the sets $A(0), \overline{A}(0), D_A, D_{\overline{A}}, D_{A^*}$, and the word "open" by the word "continuous."

Remark 2. Taking into account Theorems 1 and 2, one can obtain some assertions of Theorem 3 for an arbitrary (not necessarily closed) linear operator.

Theorem 4. Let X be a Fréchet space; M and N vector subspaces in X; M_1 and N_1 vector subspaces in X^*.

1°. If the sum $M + N$ is closed, then the sum $M^o + N^o$ is weakly closed.
2°. If the sum $M_1 + N_1$ is weakly closed, then the sum $M_1^o + N_1^o$ is closed.
3°. If X is a Banach space, then, in order that the sum $M^o + N^o$ be weakly closed, it is necessary and sufficient that it be strongly closed.

For the proof of the theorem it is enough to apply Theorem 3 to the operator A having graph

$$\Gamma_A = \Gamma_I \cap ((M, M) + (0, N)),$$

where Γ_I is the graph of the identity operator in X.

Theorem 4 is equivalent to the properties of Theorem 3 that pertain to the sets R_A and R_{A^*}.

If A and B are linear operators, then, under appropriate conditions, the operators are defined in a natural way: the sum $A + B$ and the product BA. The following theorem is proved on the basis of Theorem 3.

Theorem 5. *Let A and B be closed linear operators acting in a Fréchet space. Then:*

I. 1°. *If* $D_A + D_B$ *is closed, then* $(A+B)^* = A^* + B^*$.
2°. *If* $D_{A^*} + D_{B^*}$ *is weakly closed, then* $\Gamma_{A+B} = \Gamma^\circ_{-(A^*+B^*)}$.

II. 1°. *If* $R_A + D_B$ *is closed, then* $(BA)^* = A^* B^*$.
2°. *If* $D_{A^*} + R_{B^*}$ *is weakly closed, then* $\Gamma_{BA} = \Gamma^\circ_{-A^*B^*}$.

If the operators A *and* B *act in Banach spaces, then the sets* $D_{A^*} + D_{B^*}$ *and* $D_{A^*} + R_{B^*}$ *are weakly closed if and only if they are strongly closed.*

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REFERENCES

1. N. Bourbaki, *Topological Vector Spaces*, IL, 1959.

Note: Figure translations are in progress. See original paper for figures.