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Abstract

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ON NONDEGENERATE SPECTRA OF LOCALLY CONVEX SPACES

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In the present note two facts are reported that concern nondegenerate spectra of locally convex spaces and are related to certain questions considered in ([1]). In addition, the note establishes a necessary and sufficient condition for the perfect completeness (hypercompleteness) of the limit of a nondegenerate inverse spectrum of perfectly complete (hypercomplete) semireflexive spaces, and with the aid of this condition gives a negative answer to one question posed by Kelley.

Definition 1 ([1]). An inverse spectrum $\{X_\alpha, \pi_\alpha^\beta\}$ is called **standard** if, for all $\alpha < \beta$, $\pi_\alpha^\beta(X_\beta)$ is dense in X_α, and **nondegenerate** if the projection $\pi_\alpha(X)$ of its limit X is dense in X_α for every α.

Definition 1′ ([1]). A direct spectrum $\{Y^\alpha, \pi_\beta^\alpha\}$ is called **standard** if all π_β^α are one-to-one (so that $\{Y^\alpha, \pi_\beta^\alpha\}$ may be regarded as a direct spectrum with embeddings), and **nondegenerate** if its limit Y is separable.

Definition 2 ([1]). The spectrum **conjugate** to the inverse spectrum $\{X_\alpha, \pi_\alpha^\beta\}$ is the direct spectrum $\{Y^\alpha, \pi_\beta^\alpha\}$, where $Y^\alpha = X'_\alpha$ is the strong dual of X_α, and $\pi_\beta^\alpha = (\pi_\alpha^\beta)^*$ is the mapping conjugate to π_α^β.

Definition 2′ ([1]). The spectrum **conjugate** to the direct spectrum $\{Y^\alpha, \pi_\beta^\alpha\}$ is the inverse spectrum $\{\hat{X}_\alpha, \hat{\pi}_\alpha^\beta\}$, where $\hat{X}_\alpha = (Y^\alpha)'$ is the strong dual of Y^α, and $\hat{\pi}_\alpha^\beta = (\pi_\beta^\alpha)^*$ is the mapping conjugate to π_β^α.

Theorem 1. If the inverse spectrum $\{\hat{X}_\alpha, \hat{\pi}_\alpha^\beta\}$, conjugate to the standard direct spectrum $\{Y^\alpha, \pi_\beta^\alpha\}$, is nondegenerate, then the spectrum $\{Y^\alpha, \pi_\beta^\alpha\}$ is also nondegenerate.

Theorem 2. The inverse spectrum $\{\hat{X}_\alpha, \hat{\pi}_\alpha^\beta\}$, conjugate to a standard nondegenerate direct spectrum $\{Y^\alpha, \pi_\beta^\alpha\}$ of semireflexive spaces Y^α, is nondegenerate.

Definition 3 ([2]). A separable locally convex space X is called **perfectly complete** if, in its dual X', every weakly closed subspace M is weakly closed provided it has weakly closed intersections $M \cap U^{X'}$ with the polars $U^{X'}$ of all neighborhoods of zero U of the space X.*

Definition 4 ([4]). A separable locally convex space X is called **hypercomplete** if, in its dual X', weakly closed—

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* Such spaces, under the name hereditarily B-complete, were first considered by Pták ([3]).

then every circled convex set M, having weakly closed intersections $M \cap U^{X'}$ with the polars $U^{X'}$ of all neighborhoods U of zero in the space X.

Theorem 3. Let X be the limit of a nondegenerate inverse spectrum $\{X_\alpha, \pi_\alpha^\beta\}$ of complete (hypercomplete) semireflexive spaces X_α. Denote by Y the limit of the direct spectrum $\{Y^\alpha, \pi_\beta^\alpha\}$ conjugate to the spectrum $\{X_\alpha, \pi_\alpha^\beta\}$. In order that the space X be complete (hypercomplete), it is necessary and sufficient that every subspace (circled convex set) $M \subset Y$ for which $\overline{\pi^\alpha}(M)$ is closed in Y^α for each α, be closed in Y.

Proof. Necessity. Let X be complete (hypercomplete), and let M be a subspace (circled convex set) in Y such that $\overline{\pi^\alpha}(M)$ is closed in Y^α for each α. The space conjugate to X is algebraically isomorphic to Y ([5]). If we prove that the sets $M \cap U^Y$, where U runs through a fundamental system of neighborhoods of zero in X, are $\sigma(Y, X)$-closed, then from this, by the assumption, it will follow that M is $\sigma(Y, X)$-closed, and hence closed in Y. Thus it suffices to prove that the sets $M \cap U^Y$ are $\sigma(Y, X)$-closed.

Sets of the form $\overline{\pi_\alpha}^{-1}(U_\alpha)$, where U_α runs through a fundamental system of neighborhoods of zero in X_α, form a fundamental system of neighborhoods of zero in X. From the nondegeneracy of the spectrum $\{X_\alpha, \pi_\alpha^\beta\}$ there follows the equality

$$[\overline{\pi_\alpha}^{-1}(U_\alpha)]^Y = \pi^\alpha(U_\alpha^{Y^\alpha}).$$

Further, we have

$$M \cap U^Y = M \cap [\overline{\pi_\alpha}^{-1} U_\alpha]^Y = M \cap \pi^\alpha(U_\alpha^{Y^\alpha})$$

and

$$\overline{\pi^\alpha}(M \cap U^Y) = \overline{\pi^\alpha}(M) \cap \overline{\pi^\alpha}[\pi^\alpha(U_\alpha^{Y^\alpha})].$$

From the nondegeneracy of the inverse spectrum $\{X_\alpha, \pi_\alpha^\beta\}$ its standardness follows ([1]), and then all π^α are one-to-one by virtue of the standardness of the direct spectrum $\{Y^\alpha, \pi_\beta^\alpha\}$, conjugate to the standard inverse spectrum $\{X_\alpha, \pi_\alpha^\beta\}$ ([1]). From the one-to-one nature of π^α follows the equality

$$\overline{\pi^\alpha}[\pi^\alpha(U_\alpha^{Y^\alpha})] = U_\alpha^{Y^\alpha}.$$

Finally we have

$$\overline{\pi^\alpha}(M \cap U^Y) = \overline{\pi^\alpha}(M) \cap U_\alpha^{Y^\alpha},$$

whence it is clear that $\overline{\pi^\alpha}(M \cap U^Y)$ is closed in Y^α, and therefore, since $\overline{\pi^\alpha}(M \cap U^Y)$ is convex and X_α is semireflexive, it is $\sigma(Y^\alpha, X_\alpha)$-closed. Thus, $\overline{\pi^\alpha}(M \cap U^Y)$

is $\sigma(Y^\alpha, X_\alpha)$-closed and is contained in the $\sigma(Y^\alpha, X_\alpha)$-bicompact set $U_\alpha^{Y^\alpha}$; consequently it is itself $\sigma(Y^\alpha, X_\alpha)$-bicompact. Then $M \cap U^Y$ is $\sigma(Y, X)$-bicompact as the image of the $\sigma(Y^\alpha, X_\alpha)$-bicompact set $\overline{\pi^\alpha}(M \cap U^Y)$ under the continuous mapping π^α from the space $(Y^\alpha, \sigma(Y^\alpha, X_\alpha))$ into the space $(Y, \sigma(Y, X))$. Thus, necessity is proved.

Sufficiency. Suppose that every subspace (circled convex set) $M \subset Y$ for which $\overline{\pi^\alpha}(M)$ is closed in Y^α for each α, is closed in Y, and prove that X is complete (hypercomplete). Let M be a subspace (circled convex set) in Y, having $\sigma(Y, X)$-closed intersections $M \cap U^Y$ with the polars U^Y of all neighborhoods U of zero of the space X. We have

$$M \cap \pi^\alpha(U_\alpha^{Y^\alpha}) = M \cap [\overline{\pi_\alpha}^{-1}(U_\alpha)]^Y,$$

hence $M \cap \pi^\alpha(U_\alpha^{Y^\alpha})$ is $\sigma(Y, X)$-closed. Then, since π^α is one-to-one,

$$\overline{\pi^\alpha}^{-1}(M) \cap U_\alpha^{Y^\alpha}$$

is $\sigma(Y^\alpha, X_\alpha)$-closed as the preimage of the closed set $M \cap \pi^\alpha(U_\alpha^{Y^\alpha})$ under the continuous mapping π^α from the space $(Y^\alpha, \sigma(Y^\alpha, X_\alpha))$ into the space $(Y, \sigma(Y, X))$. By virtue of the completeness (hypercompleteness) of X_α, from the $\sigma(Y^\alpha, X_\alpha)$-closedness of the sets

$$\overline{\pi^\alpha}^{-1}(M) \cap U_\alpha^{Y^\alpha}$$

it follows that $\overline{\pi^\alpha}^{-1}(M)$

is $\sigma(Y^\alpha, X_\alpha)$-closed, and hence closed in Y^α. Since this is true for all α, it follows, by assumption, that M is closed in Y. From the semireflexivity of all X_α it follows that the conjugate of Y is algebraically isomorphic to X ([5]), and then from the closedness of the convex set M it follows that it is $\sigma(Y, X)$-closed; but this means that X is complete (hypercomplete).

Corollary. *In order that the strong dual of a reflexive LF-space E be complete (hypercomplete), it is necessary and sufficient that in E every subspace (circled convex set) M having closed intersections $M \cap E^n$ with all subspaces of the defining sequence (E^n) be closed.*

Proof. In the dual E' of the LF-space E, the strong topology coincides with the topology of the limit of the inverse spectrum $\{E_n, \varphi_n^m\}$ dual to the direct spectrum $\{E^n\}$ with embeddings. It is easy to see that the spectrum $\{E_n, \varphi_n^m\}$ is nondegenerate. Since the reflexivity of E implies the reflexivity of all E^n, their strong duals E_n are hypercomplete as reflexive DF-spaces*, and the assertion of the corollary is obtained from Theorem 3, applied to the limit of the inverse spectrum $\{E_n, \varphi_n^m\}$.

Remark. As D. A. Raikov observed, the corollary proved above can also be formulated as follows: *in order that the strong dual of a reflexive LF-space E be complete (hypercomplete), it is necessary and sufficient that in E every subspace (circled convex set) M containing the limits of all its convergent sequences be closed.*

Kelley in ([4]) expressed the supposition that the product and the direct sum of a countable family of hypercomplete spaces are hypercomplete. We shall show that *the product of a sequence of complete (hypercomplete) spaces may fail to be a complete (hypercomplete) space.*

Let E be a reflexive LF-space and let there exist in E a nonclosed subspace M having closed intersections $M \cap E^n$ with all spaces of the defining sequence (E^n)**. Consider the inverse spectrum $\{E_n, \varphi_n^m\}$ dual to the spectrum with embeddings $\{E^n\}$.

All E_n are hypercomplete (and hence complete) as reflexive DF-spaces. The limit of the spectrum $\{E_n, \varphi_n^m\}$ is closed in the product of the spaces E_n and, being isomorphic to the strong dual of the space E, is not complete by virtue of the corollary to Theorem 3. Therefore, since closed subspaces of complete spaces are complete ([2]), the product of the spaces E_n is not complete (and hence not hypercomplete).

Remark. As D. A. Raikov noted in his review of Collins' s paper ([2]) (see ([8])), *the direct sum of a sequence of complete (hypercomplete) spaces may fail to be a complete (hypercomplete) space.*

Indeed, let E be an incomplete LF-space***, (E^n) its defining sequence. The spaces E^n are hypercomplete (and hence complete) as F-spaces. From the separability of E it follows that it is isomorphic to the quotient space, by a closed subspace, of the direct sum of the spaces E^n. Since quotient spaces by closed subspaces of complete spaces

* The proof of hypercompleteness of a reflexive DF-space can be carried out according to the plan of the proof of Theorem 6.5 in ([6]).

** Grothendieck showed (see ([7]), § 2) that such spaces exist.

*** As follows from one example of Grothendieck (see ([7]), § 2), such spaces exist.

are perfectly complete ([2,3]), while E is not perfectly complete, then the direct sum of the spaces E^n is not perfectly complete (and hence is not hypercomplete either).

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Note: Figure translations are in progress. See original paper for figures.