MATHEMATICS

A. V. CHERNAVSKII

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. V. CHERNAVSKII

ON TWO-FOLD CONTINUOUS DECOMPOSITIONS OF A BALL

(Presented by Academician P. S. Aleksandrov, December 25, 1959)

1. This article is devoted to the proof, in the case $n \leqslant 3$, of the following proposition:

Theorem. *In every continuous decomposition of the ball* Q^n $(n \leqslant 3)$ *into pairs of points, there exists an element of the decomposition that degenerates into a point.*

The argument is carried out for the case Q^3, but it is applicable also to lower dimensions, and, with minor changes, also to Euclidean spaces E^n, $n \leqslant 3$. For $n = 1$ the proof was given by Harrold (^{3}), and for $n = 2$ by Roberts (^{4}).

2. Suppose that a two-fold continuous decomposition of some space is given. We shall call points that belong to one element of the decomposition **conjugate**, and denote the point conjugate to a point x by $\bar{x}$. The continuity of the decomposition means that if a point y is sufficiently close to a point x, then the conjugate point $\bar{y}$ lies either close to x or close to $\bar{x}$.

3. The following theorem of Newman (Theorem 2 of (^{1})) will be used in a form that we shall call the local case of this theorem.

Newman' s theorem. *The set of fixed points of a uniformly continuous involution of a connected locally Euclidean space is nowhere dense or coincides with the whole space.*

Local case of Newman' s theorem. *If, in a locally Euclidean space E, a topological mapping f of a neighborhood O of a certain point x into the same space is given, and if f has period 2 on $O \cup f(O)$ and leaves the point x fixed, then in some neighborhood B of the point x the set of fixed points of f is nowhere dense or fills it.*

For the proof, take a spherical neighborhood H of the point x such that

$$[H] \cup [f(H)] \subset O \cap f(O).$$

We shall denote by $[M]_X$, $\mathrm{Int}_X(M)$, $\mathrm{fr}_X(M)$, respectively, the closure, the open kernel, and the boundary of the set M in X.

As the required neighborhood of the point x it is enough, by Newman's theorem, to take $B = H \cup f(H)$, since the second summand is the topological image of the first, both are open, connected, the intersection contains the point x and is nonempty, and f is continuous on $[B]$.

4. **Preliminary construction.** Let Q^3 be a ball situated in Euclidean space E^3 with boundary S^2. Suppose that there exists a continuous decomposition φ of the ball strictly into pairs of points. Denote by M_n the closed set of points x for which $d(x, \bar{x}) \geq \frac{1}{n}$ (d is the distance between points). By assumption,

$$Q^3 = \bigcup_{n=1}^{\infty} M_n.$$

Now denote Q^3 by P_0, and the sum of the open kernels of M_n in P_0 by H_0. Since P_0 is complete, by the well-known Baire property, H_0 is everywhere dense in P_0.

$P_1 = P_0 \setminus H_0$ is closed and nowhere dense in P_0. Moreover,

$$P_1 = \bigcup_{n=1}^{\infty} (M_n \cap P_1).$$

Put $H_1 = \bigcup_{n=1}^{\infty} \mathrm{Int}_{P_1}(M_n \cap P_1)$ and $P_2 = P_1 \setminus H_1$. P_2 is closed, nowhere dense in P_1, and

$$P_2 = \bigcup_{n=1}^{\infty} (M_n \cap P_2).$$

The process is naturally continued by induction and gives us a strictly decreasing sequence of closed sets P_α and sets H_α. There is a $\nu < \omega_1$ such that P_ν is empty. Hence, Q^3 is the sum of the disjoint sets H_α, $0 \leq \alpha < \nu$.

5. Lemmas.

Lemma 1. P_1 is the set of points in every neighborhood of which there are conjugate pairs, and H_0 is the set of points possessing neighborhoods that contain no conjugate pairs.

We prove the second part of the lemma. If x lies in the open core M_n, then near x there are no conjugate pairs. If, however, x has a neighborhood without conjugate pairs, then the points conjugate to the points of a small neighborhood of x must lie in a small neighborhood of $\bar{x}$, and therefore, if $\frac{1}{n} < d(x, \bar{x})$, then x lies with a neighborhood in M_n, i.e. $x \in H_0$.

Lemma 2. The correspondence $x \to \bar{x}$ is a homeomorphism in a neighborhood of x, if $x \in H_0$.

By the preceding lemma, the continuity of the decomposition φ, and the openness of the set H_0, an entire neighborhood of x is carried by this correspondence into a neighborhood of $\bar{x}$.

Corollary. If an interior point of the ball $x \in H_0$, then $\bar{x}$ also lies inside the ball and belongs to H_0.

It suffices to apply Lemma 2 to a spherical neighborhood of the point x.

Lemma 3. If $x \in P_1$, then arbitrarily close to x there are conjugate pairs belonging to H_0.

By construction, H_0 is everywhere dense in Q^3. If, for all points of H_0 near x, the conjugate points lay near $\bar{x}$, then, by the continuity of φ, in general all points sufficiently close to x would have their conjugates near $\bar{x}$, i.e. x would lie in H_0.

6. Main lemma. The mapping of the ball onto itself, defined as follows:

$$e(x) = x, \quad \text{if } x \in P_1, \tag{1}$$

$$e(x) = \bar{x}, \quad \text{if } x \in H_0, \tag{2}$$

is a continuous involution of the ball.

The proof breaks up into three stages: first it is proved that e is continuous at the interior points of the ball; then that e is an involution; and, finally, that e is continuous at the points S^2.

The proof of the continuity of e at interior points is carried out by induction for the sets H_α. Its continuity at the points H_0 follows from Lemma 2. Suppose e is continuous at the points of all H_α for $\alpha < \beta$, and suppose $x \in H_\beta$. We shall show that e is also continuous at the point x.

Surround x and $\bar{x}$ by ε-neighborhoods, putting

$$0 < \varepsilon < \min\left(\frac{1}{3} d(x, \bar{x}),\ \frac{1}{2} d(x,\ P_{\beta+1} \cup S^2)\right). \tag{3}$$

Then

$$[O_\varepsilon(x)] \cap [O_\varepsilon(\bar{x})] = \Lambda, \tag{4}$$

$$[O_\varepsilon(x)] \cap (H_\gamma \cup S^2) = \Lambda \quad \text{for } \gamma > \beta. \tag{5}$$

By the continuity of the partition φ, choose $\delta_1 > 0$ so that if $y \in O_{\delta_1}(x)$, then $\bar{y} \in O_\varepsilon(x \cup \bar{x})$. By the construction of the set H_β, since $x \in H_\beta$, one can find $\delta_2 > 0$ such that $y \in O_\varepsilon(\bar{x})$ if $y \in H_\beta \cap O_{\delta_2}(x)$.

Let $0 < \delta < \min(\varepsilon, \delta_1, \delta_2)$. Consider e in $O_\delta(x)$. It is fixed at the points of all H_α for $\alpha > 0$, and hence its continuity at the points of H_β can fail only because of points of H_0 whose conjugates lie in $O_\varepsilon(\bar{x})$. Denote the set of such points by Φ. We shall show that Φ is empty near x. It is, obviously, open and

$$\mathrm{fr}_{O_\delta} \Phi \subset H_\beta, \tag{6}$$

since at a boundary point of Φ not lying in H_β, the continuity of e, which by induction we assume everywhere except H_β, would be violated. Define a mapping g on O_δ by setting

$$g(y) = y, \qquad \text{if } y \in \Phi; \tag{7}$$

$$g(y) = e(y), \qquad \text{if } y \in \overline{\Phi}. \tag{8}$$

We shall show that g is continuous on O_δ. This is clear for the points of H_0 in O_δ. For the points H_α, $0 < \alpha < \beta$, it follows from (6), (8), and the continuity of e at these points. Let $y \in H_\beta$ and let y' be a point close to y. If y' belongs to Φ or to H_α, $0 < \alpha \leq \beta$, then y' is fixed, like y. If, however, $y' \in H_0 \setminus \Phi$, then $g(y') = \bar{y}'$, and $g(y')$ remains near y. Thus g is continuous at all points of O_δ. The hypotheses of the local case of Newman' s theorem are satisfied. By Lemma 3, Φ does not exhaust all the points of H_0 near the point x. Therefore the set of fixed points of the mapping g does not contain a neighborhood of x, and hence there exists a neighborhood B in which the set of fixed points of g is nowhere dense. Since Φ is open, Φ is empty in B. Consequently, g coincides in B with e, and e is continuous at x, as is g. Thus, whatever α may be, e is continuous at the points of H_α, and hence everywhere inside the ball.

We now show that e is an involution. By virtue of the corollary to Lemma 2, it is enough to show that if $x \in H_0 \cap S^2$, then $\bar{x} \in H_0$. First we shall show that $\bar{x} \in S^2$. If $\bar{x} \in Q^3/S^2$, then, by the corollary to Lemma 2, $\bar{x} \in P_1$. By Lemma 3, near $\bar{x}$ there are conjugate pairs from H_0. On the other hand, one can find in H_0 a conjugate pair y and $\bar{y}$ such that y lies near x, and $\bar{y}$ near $\bar{x}$. Hence e has a discontinuity at the point $\bar{x}$, which is impossible at an interior point. Thus $\bar{x}$ lies on S^2. We shall show that $\bar{x} \in H_0$. By what has just been proved and by Lemma 2, the correspondence $y \to \bar{y}$ topologically maps a neighborhood $O_\varepsilon(x) \subset H_0$ so that the points of S^2 go over into S^2, and interior points into interior points. Consequently, $e(O_\varepsilon)$ is a neighborhood of the point $\bar{x}$, having no conjugate pairs if O_ε has none. By Lemma 1, $\bar{x} \in H_0$.

It follows from this argument that e is continuous at the points of H_0. It remains to show that e is continuous at the points of $P_1 \cap S^2$. Let $x \in P_1 \cap S^2$. By Lemma 3, in any neighborhood of x there lie conjugate pairs belonging to H_0. If e is discontinuous at x, then arbitrarily close to x there lie points of H_0 whose conjugates are near $\bar{x}$. The set Φ of such points is separated in an arbitrarily small neighborhood of x from $H_0 \setminus \Phi$ by the set P_1. Arbitrarily close to x one can take a point $y \in P_1 \cap [\Phi] \cap \operatorname{Int} Q^3$, and e will be discontinuous at y, which is impossible, since y lies inside the ball.

7. Proof of the theorem. We have shown that e is a continuous involution of the ball. Moreover, it has been shown that if $x \in H_0 \cap S^2$, then

and $\bar{x} \in H_0 \cap S^2$. This makes it possible to extend e outside Q^3 in the following way. Through each point $x \in S^2$ draw the ray $r(x)$ from the center of the ball, and, if $x \in H_0$, assign to each exterior point of this ray the point of the ray $r(\bar{x})$ equidistant from the center. If, however, $x \in P_1$, leave the exterior points of the ray $r(x)$ fixed.

Next adjoin to E^3 a fixed point p up to the sphere S^3. It is clear that we obtain a continuous involution on S^3. The set of fixed points of a continuous involution of the three-dimensional sphere, as Smith has shown (^{2}, p. 707), is homeomorphic to a sphere of smaller dimension. The possible cases are: $-1, 0, 1, 2$. The first is excluded, since at least p is fixed. In the second case—the zero-dimensional sphere, i.e., a pair of points—P_1 contains one point which has no conjugate, since x and $\bar{x}$ belong to P_1 or to H_0 simultaneously. This contradicts the assumption.

The last two cases reduce to the preceding one. Indeed, let K be the set of fixed points of e in S^3. Then $K \setminus Q^3$ is a family of rays issuing from the point p and filling its neighborhood with boundary lying on S^2. In the first case such a family fills a simple arc, and in the second a 2-cell. Since K is a sphere, P_1 is likewise either a simple arc or a 2-cell, i.e., either Q^1 or Q^2. Moreover, φ induces on P_1 a strictly double continuous partition. In this case the arguments carried out for Q^3 are applicable to P_1. Repeating them once or twice, we arrive at a point which, contrary to the assumption, has no conjugate point. The proof is complete.

The author expresses his sincere gratitude to Prof. L. V. Keldysh for her attention to this work.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
24 XI 1959

CITED LITERATURE

[1] M. H. A. Newman, Quart. J. Math., Oxford ser., **2**, 1 (1931).
[2] P. A. Smith, Ann. Math., Second ser., **40**, No. 3, 690 (1939).
[3] O. G. Harrold, Duke Math. J., **5**, 789 (1939).
[4] J. H. Roberts, Duke Math. J., **6**, 256 (1940).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.