ON THE GROUP OF DIVISOR CLASSES ON THE CURVE \(x^4+y^4=1\)

1960

SovietRxiv

Abstract

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MATHEMATICS

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ON THE GROUP OF DIVISOR CLASSES ON THE CURVE $x^4 + y^4 = 1$

(Presented by Academician I. M. Vinogradov on 19 V 1960)

The aim of the present note is to prove the finiteness of the group of divisor classes of degree zero for the field of algebraic functions $k = R(x, y)$, $x^4 + y^4 = 1$, where R is the field of rational numbers.

1°. Let S be the field of all complex numbers. The field $K = S(x, y)$, $x^4 + y^4 = 1$, has genus $g = 3$. Among its subfields the field K contains the following: $K_1 = S(\xi, y)$, $\xi = x^2$, $\xi^2 = 1 - y^4$; $K_2 = S(x, \eta)$, $\eta = y^2$, $\eta^2 = 1 - x^4$; $K_3 = S(u, v)$, $u = x/y$, $v = 1/y^2$, $v^2 = u^4 + 1$. The genus of each of the fields K_i is equal to one.

Denote, respectively, by C, C_1, C_2, C_3 the Riemann surfaces for the fields K, K_1, K_2, K_3. The surface C covers each of the surfaces C_i twice. By ν_i denote the norm mappings of points of the surface C onto the surface C_i, which assign to each point $p \in C$ the points on C_i that are covered by the point p. The mappings ν_1, ν_2, ν_3 extend naturally to the divisor group, to the group of divisors of degree zero, and to the divisor class group of the field K, and give homomorphic mappings of these groups into the corresponding groups of the fields K_i. On the other hand, the mappings ν_i extend to the groups of 1-chains, 1-cycles, and 1-homologies of the surface C, mapping them into the groups of the same name on the surfaces C_i. Denote by ν the mapping of the Jacobian variety J (i.e. the group of divisor classes of degree zero) of the field K into the direct product $J_1 \times J_2 \times J_3$ of the Jacobian varieties of the fields K_i, acting by the formula $\nu(a) = \nu_1(a) \times \nu_2(a) \times \nu_3(a)$, $a \in J$, and by ν^* the mapping of the one-dimensional homology group H of the surface C into the direct sum $H_1 \oplus H_2 \oplus H_3$ of the groups of 1-homologies of the surfaces C_i, acting by the formula $\nu^*(b) = \nu_1(b) \oplus \nu_2(b) \oplus \nu_3(b)$, $b \in H$. We shall show that the mapping ν^* is monomorphic with finite kernel and the mapping ν is epimorphic with finite kernel. For this purpose we introduce into consideration the differentials of the first kind $\omega_1 = du/\xi$, $\omega_2 = dx/\eta$, and $\omega_3 = du/v$ for the fields K_i. These differentials (more precisely, their traces in the field K, cf. ([1])) are, respectively, $-x\,dx/y^3$, dx/y^2, dx/y^3, and, consequently, form a complete system of linearly independent differentials of the first kind for the field K. If a 1-cycle γ of the surface C is not homologous to zero, then at least one of the integrals $\int_\gamma \cos p\,\omega_i = \int_{\nu_i(\gamma)} \omega_i$, $i = 1, 2, 3$, is different from zero, and consequently at least one of the cycles $\nu_i(\gamma)$ is not homologous to zero. Thus the mapping ν^* is monomorphic. The finiteness of the kernel follows from the fact that both the group H and the group $H_1 \oplus H_2 \oplus H_3$ are free abelian groups with the same number of generators. An actual computation shows that the kernel is a group of order 8 of type $(2, 2, 2)$. Since the group H is in a natural one-to-one correspondence with the period lattice of integrals of the first kind, the mapping ν^* induces a mapping of this lattice for the surface C into the direct sum of the period lattices for the surfaces C_i. The space of values of integrals of the first kind for C is the direct sum of the spaces of values of integrals for C_i. The Jacobian variety is the quotient space of the space of values by the period lattice. Therefore ν is epi-

morphically maps J to $J_1 \times J_2 \times J_3$, and the kernel is a group of order 8 of type $(2, 2, 2)$.

2°. Let now R be the field of rational numbers and $k = R(x, y)$, $x^4 + y^4 = 1$. The mapping v, obviously, maps the class group of rational divisors of degree zero of the field k into the direct product of the same groups for the fields $k_1 = R(\xi, y)$, $k_2 = R(x, \eta)$ and $k_3 = R(u, v)$, and this mapping is no longer obliged to be an epimorphism. But in fields of genus 1 the class group of rational divisors of degree zero is isomorphic to the group of rational points, and Euler also proved that on the curves $\xi^2 = 1 - y^4$ and $v^2 = 1 + u^4$ there are no rational points except for the four trivial ones. Since the kernel of the mapping v is finite, it follows from this that the class group of rational divisors of degree zero on the curve $x^4 + y^4 = 1$ is finite. The order of this group does not exceed $8 \cdot 4^3 = 2^9$. In fact the order turns out to be equal to 32.

3°. Let us establish some consequences. From the theorem proved it follows that the number of classes of rational divisors of any fixed degree is equal to 32. The dimension of each class of degree two is equal to 0 or 1, since the curve $x^4 + y^4 = 1$ is not hyperelliptic. Consequently, there exist no more than 32 integral rational divisors of degree two. Therefore there exists only a finite number of quadratic fields in which the equation $x^4 + y^4 = 1$ has solutions not belonging to R, and in each of them the equation has only a finite number of solutions. Such fields are only $R(\sqrt{-1})$ and $R(\sqrt{-7})$, and the solutions in them are

$$(\pm i, 0); \quad (0, \pm i); \quad \left(\sigma_1 \frac{1 + \sigma_3\sqrt{-7}}{2},\ \sigma_2 \frac{1 - \sigma_3\sqrt{-7}}{2}\right), \quad \sigma_1 = \pm 1,\ \sigma_2 = \pm 1,\ \sigma_3 = \pm 1.$$

According to the Riemann-Roch theorem, classes of degree 3 have dimension 1 or 2, with dimension 2 possible only for classes containing integral divisors which are divisors of integral divisors of the canonical class. These latter are quadruples of points lying on one straight line. 28 of the 32 classes of degree 3 have dimension 1, and the integral divisors contained in them decompose into products of integral rational divisors of smaller degree. The classes of dimension 2 are formed by triples of points on four pencils of straight lines passing through the rational points of the curve $x^4+y^4=1$. Therefore the equation $x^4+y^4=1$ has no solutions in cubic extensions of the field R, with the exception of the points of intersection of the curve with the straight lines of the indicated four pencils.

4°. The result on the finiteness of the group of classes of divisors of degree zero on the curve $x^4+y^4=1$ remains valid if the field R is replaced by the field $R_1=R(\zeta)$, $\zeta^4=-1$, with all the consequences following from this. The number of exceptional quadratic and cubic extensions of the field R_1 in which the equation $x^4+y^4=1$ is solvable in a nontrivial way becomes somewhat larger than for the field R.

5°. Let us note that the Jacobian variety for the Fermat field $K=S(x,y)$, $x^p+y^p=1$, whose genus is $\frac{1}{2}(p-1)(p-2)$, also admits a mapping with finite kernel to the Jacobian varieties of fields of smaller genus, namely of the fields $K_i=S(u,v)$, $u=x^p$, $v=x^iy$, $v^p=u^i(1-u)$, $i=1,\dots,p-2$, the genus of each of which is equal to $\frac{1}{2}(p-1)$, which follows from the linear independence of the cosets of differentials of the first kind for the fields K_i.

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Received
6 V 1960

CITED LITERATURE

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2. L. Euler, *Opera omnia*, Leipzig–Berlin, Ser. 1–3, 1911–1955.

Note: Figure translations are in progress. See original paper for figures.