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Abstract

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ON THE ISOMORPHISM AND APPROXIMATIVE DIMENSION OF SPACES OF HOLOMORPHIC FUNCTIONS

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In this note we present certain results concerning the isomorphic classification and the classification by approximative dimension of various spaces of holomorphic functions of many variables.

Let X, Y be linear-metric spaces. The spaces X and Y are called **isomorphic** if there exists a linear operator U mapping X one-to-one onto the space Y, continuous together with its inverse.

Let X be a linear-metric space. Denote by $\Phi(X)$ the class of functions $\varphi(\varepsilon)$ such that, for every compact set K and every neighborhood of zero U, there exists a number $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ there exist $N \leqslant \varphi(\varepsilon)$ points $x_1, \dots, x_N$ such that*

$$K \subset \bigcup_{n=1}^{N} (x_n + \varepsilon U).$$

The spaces X and Y have equal approximative dimension, $d_a(X) = d_a(Y)$, if $\Phi(X) = \Phi(Y)$. The space X has approximative dimension no greater than that of the space Y, $d_a(X) \leqslant d_a(Y)$, if $\Phi(X) \supset \Phi(Y)$. The approximative dimension of the space X is less than the approximative dimension of the space Y, $d_a(X) < d_a(Y)$, if $d_a(X) \leqslant d_a(Y)$ and $d_a(X) \neq d_a(Y)$. If the spaces X and Y are isomorphic, then $d_a(X) = d_a(Y)$. If X is a subspace of Y, then $d_a(X) \leqslant d_a(Y)$. The concept of approximative dimension belongs to A. N. Kolmogorov ([5]).

Let D be a domain of k-dimensional complex space. By $\hat{H}(D)$ we denote the space of all holomorphic functions $x(z_1, \dots, z_k)$ defined in the domain D, with the usual operations of addition and multiplication by a complex number and with the locally convex topology induced by all seminorms

$$\|x\|_F = \sup_{(z_1,\dots,z_k)\in F} |x(z_1, \dots, z_k)|,$$

where F is an arbitrary compact set contained in D. There exists a metric equivalent to this topology such that the space is a complete linear-metric space.

Convergence of a sequence in the space $\hat{H}(D)$ is equivalent to almost uniform convergence. For every D the space $\hat{H}(D)$ is a nuclear space.**

In this note we investigate isomorphisms and equalities or inequalities of the approximative dimension of the space $H(D)$ for various classes of domains D. In addition, for some classes of domains D there is given

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* $(x_n + \varepsilon U)$ denotes the set of all elements of the form $x_n + \varepsilon y$, where $y \in U$.

** The definition and properties of nuclear spaces are given in ([4]); see also ([9]).

an isomorphic representation of the space $H(D)$ by means of Köthe sequence spaces. The note contains a generalization of some results of the works ([4–6,8]).

A domain D is called a **polycylinder** if it is the Cartesian product of one-dimensional domains D_i, $i = 1, 2, \dots, k$, $D = D_1 \times D_2 \times \cdots \times D_k$. We shall denote by C the complex plane, by C_0 the unit disk, and by C^k (respectively C_0^k) the k-dimensional polycylinder in which $D_i = C$ (respectively C_0). Let $\{a_{m,n}\}$ be a matrix of nonnegative numbers. By $M(a_{m,n})$ we denote the space of all sequences of complex numbers $x = \{t_n\}$ such that

$$\|x\|_m = \sup_n |t_n| a_{m,n} < +\infty,$$

with convergence defined by the sequence of seminorms $\|x\|_m$*.

If in the nuclear space X there exists an unconditional basis e_n**, then this space is isomorphic to the space $M(\|e_n\|_m)$(([10]), Theorem 3). Theorems 1 and 2 are special cases of Theorem 3 ([10]).

Theorem 1. $H(C^k)$ is isomorphic to $M(2^{m \sqrt[k]{n}})$.

Theorem 2. Let D be an arbitrary bounded complete circular domain*** in k-dimensional space. Then $H(D)$ is isomorphic to $M(2^{-\sqrt[k]{n/m}})$.

Theorem 3. Let D be a polycylinder, $D = D_1 \times D_2 \times \cdots \times D_k$, such that: a) all domains D_i are finitely connected; b) for $i = 1, 2, \dots, p$ the complement of D_i consists only of isolated points; c) for $i = p+1, p+2, \dots, p+r$ the complement of D_i consists only of continua; d) for $i = p+r+1, \dots, k$ the complement of D_i contains both continua and isolated points.

Then the space $H(D)$ is isomorphic to the space

$$H(C^p \times C_0^{k-p}) \times H(C^{p+1} \times C_0^{k-p-1}) \times \cdots \times H(C^{k-r} \times C_0^r).$$

Theorem 4. Let D be a k-dimensional polycylinder. Then the space $H(D)$ is isomorphic to some subspace of $H(C_0^k)$. Moreover, if D is a bounded set, then $H(C_0^k)$ is also isomorphic to some subspace of $H(D)$. This means that the spaces $H(D)$ and $H(C_0^k)$ have one and the same linear dimension in the sense of Banach****.

Theorem 5. For complete circular polycylinders the relations

$$d_a(H(C_0^k)) = d_a(H(C^{k-1} \times C_0)) = d_a(H(C^{k-2} \times C_0^2)) = \cdots$$

$$\cdots = d_a(H(C \times C_0^{k-1})) < d_a(H(C_0^k)) < d_a(H(C^{k+1})).$$

hold.

Theorem 6. For any k-dimensional open set D,

$$d_a(H(C^k)) \leqslant d_a(H(D)) \leqslant d_a(H(C_0^k)).$$

* If these spaces are nuclear, then they are isomorphic to the corresponding Köthe spaces ([7]).

** The definition and properties of an unconditional basis are given in ([2]), pp. 73–77.

*** A domain D is called a **complete circular domain** with center $(a_1, \dots, a_k)$ if, from the fact that $(z_1, \dots, z_k) \in D$, it follows that $(a_1 + r(z_1 - a_1)e^{i\varphi}, \dots, a_k + r(z_k - a_k)e^{i\varphi}) \in D$ for all real φ and $|r| \leqslant 1$ ([3]), p. 114.

**** For the definition of linear dimension in the sense of Banach, see ([1]), p. 193.

From Theorems 5 and 6 it follows:

Theorem 7. *Let D be an open k-dimensional set, and D' an open k'-dimensional set. If $k < k'$, then*

$$d_a(H(D)) < d_a(H(D')).$$

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Note: Figure translations are in progress. See original paper for figures.