ON THE CONVERGENCE OF LINEAR POSITIVE OPERATORS IN THE SPACE OF CONTINUOUS FUNCTIONS

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Abstract

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ON THE CONVERGENCE OF LINEAR POSITIVE OPERATORS IN THE SPACE OF CONTINUOUS FUNCTIONS

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Consider, on some compact set Q, the space $C(Q)$ of real continuous functions $f(x)$, $x \in Q$, with norm

$$\|f\|_C = \max_{x \in Q} |f(x)|$$

and the space $B(Q)$ of real bounded functions $\varphi(x)$, $x \in Q$, with norm

$$\|\varphi\|_B = \sup_{x \in Q} |\varphi(x)|.$$

It is obvious that $C(Q)$ is a subspace of $B(Q)$. Let L_n $(n = 1, 2, \ldots)$ be a sequence of linear positive operators mapping $C(Q)$ into $B(Q)$. (An operator L is called positive if, for every nonnegative function $f(x) \in C(Q)$, the function $\varphi(x) = L(f, x) \in B(Q)$ is also nonnegative.)

Definition 1. A system of continuous functions

$$S_m = \{f_0(x), f_1(x), \ldots, f_m(x)\}$$

will be called a K-system on the compact set Q if, for any sequence of linear positive operators L_n, from

$$\|L_n(f_i, x) - f_i(x)\|_B \to 0 \qquad (i = 0, 1, 2, \ldots, m)$$

as $n \to \infty$, it follows that

$$\|L_n(f, x) - f(x)\|_B \to 0$$

for every $f(x) \in C(Q)$. The number m is called the order of the system S_m.

The properties of K-systems have been studied by P. P. Korovkin ([1,2]) (see also ([3])) in the case when the compact set Q is an interval or a circle; by V. I. Volkov ([4,5]) in the case of a closed domain in the plane; and by E. N. Morozov ([6]) in the case of the two-dimensional torus.

In this article we shall consider only such K-systems for which $f_0(x) \equiv 1$.

Definition 2. It is said that a system of continuous functions

$$S_m = \{f_0(x), f_1(x), \dots, f_m(x)\}$$

has Chebyshev rank equal to r ($0 \le r \le m$) on the compact set Q if, for any $m - r + 1$ distinct points $x_1, \dots, x_{m-r+1}$ of the compact set Q, the rank of the matrix

$$\begin{pmatrix} f_0(x_1) & f_1(x_1) & \dots & f_m(x_1) \\ f_0(x_2) & f_1(x_2) & \dots & f_m(x_2) \\ \dots & \dots & \dots & \dots \\ f_0(x_{m-r+1}) & f_1(x_{m-r+1}) & \dots & f_m(x_{m-r+1}) \end{pmatrix}$$

is equal to $m - r + 1$, and there exist $m - r + 2$ distinct points of the compact set for which the rank of the corresponding matrix is less than $m - r + 2$. In particular, if $r = 0$, then S_m is called a Chebyshev system. This definition of Chebyshev rank is equivalent to the original one (see, for example, ([7,8])).

Theorem 1. In order that on a compactum Q there exist at least one K-system of finite order, it is necessary and sufficient that the compactum Q have finite dimension.

Theorem 2. In order that the system $S_m = \{f_0(x) = 1,\ f_1(x), \dots, \dots, f_m(x)\}$ be a K-system on the compactum Q, it is necessary and sufficient that, for every point $x_0 \in Q$, a linear positive functional Φ in the space $C(Q)$ be uniquely determined by the conditions $\Phi(f_i) = f_i(x_0)$, $i = 0, 1, \dots, m$.

This theorem can be formulated in other terms:

Theorem 3. In order that the system S_m be a K-system on the compactum Q, it is necessary and sufficient that the mapping F of the compactum Q onto a subset M of m-dimensional Euclidean space E^m, defined by this system,

$$F(x) = (f_1(x), f_2(x), \dots, f_m(x)) \in E^m, \qquad x \in Q,$$

be homeomorphic and that one of the following mutually equivalent conditions be satisfied:

each point of the set M is an extreme point of its convex hull;

no point of the set M belongs to the convex hull of the remaining part of this set;

whatever simplex of positive dimension with vertices from the set M may be taken, all its interior points do not belong to this set.

From Theorem 3 there follows

Corollary ([5]). The order m and the Chebyshev rank r of a K-system on any compactum are related by the relation $m \ge r + 2$.

Denote by $m(Q)$ the minimal order of K-systems on the compactum Q. Naturally the question arises of computing $m(Q)$ for any finite-dimensional compactum. In this direction one can prove the following theorem.

Theorem 4. If the compactum Q is homeomorphic to an n-dimensional sphere or to a subset of it, then $m(Q) \leq n+1$. If there exists a nonempty subset $U \subset Q$, open relative to Q, such that the difference $Q \setminus U$ is not homeomorphically embedded in $(n-1)$-dimensional Euclidean space E^{n-1}, then $m(Q) \geq n+1$.

Proof. The first assertion of the theorem follows from the fact that on the n-dimensional sphere there exist systems of functions of order $n+1$ which are K-systems on the entire sphere and on any of its closed subsets, for example:

$$\begin{aligned} f_0 &= 1, \\ f_1 &= \cos x_1, \\ f_2 &= \sin x_1 \cos x_2, \\ &\dots\dots\dots\dots \\ f_n &= \sin x_1 \sin x_2 \cdots \sin x_{n-1} \cos x_n, \\ f_{n+1} &= \sin x_1 \sin x_2 \cdots \sin x_{n-1} \sin x_n \end{aligned}$$
$$(0 \leq x_i \leq \pi, \quad i = 1, 2, \dots, n-1; \qquad 0 \leq x_n \leq 2\pi).$$

The second assertion follows from a theorem of K. Borsuk ([9]).

Corollary. If Q is the closure of an open subset of the n-dimensional sphere or of n-dimensional Euclidean space, then $m(Q) = n+1$.

Theorem 5. Let Q be homeomorphic to the closure of an open and connected subset (i.e., a domain) of the n-dimensional sphere. In order that the system $S_{n+1} = \{f_0(x) = 1,\ f_1(x), \dots, f_{n+1}(x)\}$ be a K-system on the compactum Q,

necessary and sufficient that it have Chebyshev rank equal to $m(Q) - 2 = n-1$.

In the proof of this theorem the following is used.

Lemma. Let the compact set Q satisfy the conditions of Theorem 5, and let the system S_{n+1} have Chebyshev rank $n-1$ on Q. Then for every interior point $x_0 \in Q$ there exists a polynomial

$$p(x, x_0) = \sum_{i=0}^{n+1} c_i f_i(x)$$

with respect to the system S_{n+1} such that

$$p(x_0, x_0) = 0, \qquad p(x, x_0) > 0, \qquad x \in Q, \quad x \neq x_0.$$

One can verify from examples that the conditions imposed on the compact set are essential in Theorem 5; moreover, the following theorem is valid.

Theorem 6. For every disconnected compact set Q containing more than three points, there exists a system of functions of order $m = m(Q)$ and Chebyshev rank $r = m(Q) - 2$ which is not a K-system.

For the set consisting of the intervals $[0, \pi/2]$ and $[\pi, 3\pi/2]$, such a system will be, for example, the following Chebyshev system of second order:

$$f_0(x) = 1;$$

$$f_1(x) = \begin{cases} \cos x, & x \in [0, \pi/2], \\ 2 + \cos x, & x \in [\pi, 3\pi/2]; \end{cases} \qquad f_2(x) = \begin{cases} \sin x, & x \in [0, \pi/2], \\ 2 + \sin x, & x \in [\pi, 3\pi/2]. \end{cases}$$

Let us now consider K-systems in the space of periodic functions of n variables or, in other words, in the space $C(T^n)$, where T^n is the n-dimensional torus. It is not difficult to see that $m(T^n) = n + 2$ (for $n \geq 2$). It is also clear that any system of the form

$$f_0(x) = 1, \qquad f_1(x), \dots, f_{n+1}(x),$$

$$f_{n+2}(x) = f_1^2(x) + f_2^2(x) + \dots + f_{n+1}^2(x), \qquad x \in T^n,$$

where the functions $f_1(x), \dots, f_{n+1}(x)$ realize a homeomorphic embedding of the torus T^n into $(n+1)$-dimensional Euclidean space, for example:

$$f_1 = a_1 \cos x_1,$$

$$f_2 = (a_2 + a_1 \sin x_1) \cos x_2,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$f_n = (a_n + (a_{n-1} + \dots + (a_2 + a_1 \sin x_1) \sin x_2 \cdots) \sin x_{n-1}) \cos x_n,$$

$$f_{n+1} = (a_n + (a_{n-1} + \dots + (a_2 + a_1 \sin x_1) \sin x_2 \cdots) \sin x_{n-1}) \sin x_n,$$

$$\left(0 \leq x_i \leq 2\pi, \quad i = 1, 2, \dots, n; \qquad a_1 > 0, \quad a_k > \sum_{i=1}^{k-1} a_i, \quad k = 2, 3, \dots, n\right).$$

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CITED LITERATURE

1. P. P. Korovkin, *DAN*, **90**, No. 6, 961 (1953).
2. P. P. Korovkin, Scientific Notes of the Kalinin Pedagogical Institute named after M. I. Kalinin, **26**, 95 (1958).
3. P. P. Korovkin, *Linear Operators and Approximation Theory*, 1959.
4. V. I. Volkov, *DAN*, **115**, No. 1, 17 (1957).
5. V. I. Volkov, Scientific Notes of the Kalinin Pedagogical Institute named after M. I. Kalinin, **26**, 27 (1958).
6. E. N. Morozov, ibid., **26**, 129 (1958).
7. G. Sh. Rubinshtein, *DAN*, **102**, No. 3, 451 (1955).
8. A. L. Garkavi, *Izv. AN SSSR, Ser. Mat.*, **23**, No. 1 (1959).
9. K. Borsuk, *Bull. Acad. Sci. Pol.*, **5**, No. 4, 351 (1957).

Note: Figure translations are in progress. See original paper for figures.