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Abstract

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MATHEMATICS

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AN ALGORITHM FOR SOLVING A CERTAIN GENERALIZED PROBLEM OF LINEAR PROGRAMMING

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1. L. V. Kantorovich in ([1]) (where the literature is also indicated), as a mathematical model of the basic problem of production planning, considers the following problem:

Given the matrix

$$\begin{pmatrix} a_{11} & \dots & a_{1m} & b_{11} & \dots & b_{1s} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \dots & a_{nm} & b_{n1} & \dots & b_{ns} \end{pmatrix}$$

and the numbers $l_1, l_2, \dots, l_m$. It is required, in the region Ω defined by the inequalities

$$1) \quad \xi_i \geqslant 0 \quad (i = 1, \dots, n);$$

$$2) \quad \sum_{i=1}^{n} a_{ik}\xi_i \geqslant l_k \quad (k = 1, \dots, m),$$

to find $x^* = (\xi_1^*, \dots, \xi_n^*)$ such that

$$\min_{1 \le j \le s} \sum_{i=1}^{n} b_{ij}\xi_i^* = \max_{x \in \Omega} \min_{1 \le j \le s} \sum_{i=1}^{n} b_{ij}\xi_i.$$

Considering $x = (\xi_1, \dots, \xi_n)$ as a point of n-dimensional Euclidean space, the problem can be interpreted geometrically as follows.

In n-dimensional Euclidean space there are given s planes

$$\Delta_j(x) \equiv b_{1j}\xi_1 + b_{2j}\xi_2 + \dots + b_{nj}\xi_n = 0 \quad (j = 1, \dots, s), \tag{1}$$

passing through the origin, and a convex closed polyhedron Ω, situated in the positive octant and defined by the inequalities

$$\delta_k(x) \equiv a_{1k}\xi_1 + a_{2k}\xi_2 + \dots + a_{nk}\xi_k \geq l_k \quad (k = 1, \dots, m+n)^*. \tag{2}$$

It is necessary to find in the polyhedron Ω a point $x^* = (\xi_1^*, \dots, \xi_n^*)$ (the optimal point), which is weightedly the most distant from the planes (1)**, i.e., such that

$$\min_{1 \leq j \leq s} \Delta_j(x^*) = \max_{x \in \Omega} \min_{1 \leq j \leq s} \Delta_j(x). \tag{3}$$

* For convenience of notation, we include in (2) also the inequalities $\xi_i \geqslant 0$ $(i = 1, \dots, n)$ in the form $\delta_{m+i}(x) \geq l_{m+i}$ $(i = 1, \dots, n)$.

** By the weighted distance from a point x_1 to the plane $\Delta_j(x) = 0$ we mean the quantity $\Delta_j(x_1)$. In what follows we shall omit the word "weighted."

It is clear that, for an optimal point to exist, it is sufficient that the polyhedron Ω be bounded. It is also clear that the optimal point must lie on the boundary of Ω.

In this paper we present, unlike the solutions indicated in ([1]), a finite and monotone algorithm for finding the optimal point x^*, which is essentially a certain adaptation to the problem under consideration of the algorithm constructed in ([2–4]) in connection with the problem of Chebyshev approximation of functions by polynomials.

2. First take an arbitrary point $x' = (\xi_1', \dots, \xi_n') \in \Omega$ and find all planes (their number $p_1 \geqslant 1$) from (1) that are nearest to x', i.e., those for which $\min_{1 \leqslant j \leqslant s} \Delta_j(x')$ is attained, and also find all planes (their number $p_2 \geqslant 0$) from $\delta_k(x) = l_k$ $(k = 1, \dots, m+n)$ for which $\delta_k(x') = l_k$. Taking $p_1 + p_2 = p$, we shall call the point x' a p-th approximation (to the point x^*) and denote it by $x_p = (\xi_1^{(p)}, \dots, \xi_n^{(p)})$. Let, for example,

$$\Delta_{j_1}(x_p) = \dots = \Delta_{j_{p_1}}(x_p) < \Delta_j(x_p) \quad (j \neq j_1, \dots, j_{p_1}), \tag{4}$$

$$\delta_{k_1}(x_p) = l_{k_1}, \dots, \delta_{k_{p_2}} = l_{k_{p_2}}; \qquad \delta_k(x_p) > l_k \quad (k \neq k_1, \dots, k_{p_2}).$$

To construct the next approximation, we determine the direction of the relative gradient of the function $\Delta_{j_1}(x)$ at the point x_p, i.e., determine the vector $z = z_p = (\zeta_1^{(p)}, \dots, \zeta_n^{(p)})$ so that the derivative

$$\frac{d}{d\varepsilon}\Delta_{j_1}(x_p + \varepsilon z) = \Delta_{j_1}(z)$$

is greatest under the conditions:

$$1)\ \|z\| = \left(\sum_{i=1}^{n} \zeta_i^2\right)^{1/2} = \text{const};$$

$$2)\ \Delta_{j_1}(x_p + \varepsilon z) = \Delta_{j_\nu}(x_p + \varepsilon z) \quad (\nu = 2, \dots, p_1),$$

whence

$$2')\ \Delta_{j_1}(z) - \Delta_{j_\nu}(z) = 0 \quad \text{or} \quad \sum_{i=1}^{n}(b_{ij_1} - b_{ij_\nu})\zeta_i = 0 \quad (\nu = 2, \dots, p_1);$$

$$3)\ \delta_{k_\nu}(x_p + \varepsilon z) = b_{k_\nu} \quad (\nu = 1, \dots, p_2),$$

whence

$$3')\ \delta_{k_\nu}(z) = 0 \quad \text{or} \quad \sum_{i=1}^{n} a_{ik_\nu}\zeta_i = 0 \quad (\nu = 1, \dots, p_2).$$

Using the method of Lagrange multipliers, it is not difficult to find the vector z_p. We note that this vector lies both in the $(n-p_1+1)$-dimensional plane bisecting the planes $\Delta_{j_1}(x) = 0, \dots, \Delta_{j_{p_1}}(x) = 0$ (i.e., obtained as the intersection of $p_1 - 1$ planes ($(n-1)$-dimensional) that are bisector planes of the dihedral angles formed by each pair of the planes $\Delta_{j_1}(x) = 0, \dots, \Delta_{j_{p_1}}(x) = 0$, if $p_1 > 1$), and in the $(n - p_2)$-dimensional face of the boundary of Ω (if $p_2 > 0$).

From the point x_p we move in the direction z_p, increasing all distances $\Delta_{j_1}(x_p + \varepsilon z_p) = \dots = \Delta_{j_{p_1}}(x_p + \varepsilon z_p)$, until they become equal (at the smallest positive value $\varepsilon = \varepsilon'_p$) to the distance to one more (or several) of the planes $\Delta_j(x) = 0$ ($j \neq j_1, \dots, j_{p_1}$), or until the point $x_p + \varepsilon z_p$ falls (at the smallest positive value $\varepsilon = \varepsilon''_p$) on one more (or several) of the planes $\delta_k(x) = l_k$ ($k \neq k_1, \dots, k_{p_2}$). This reduces to finding the smallest positive $\varepsilon = \varepsilon_p$ from

of the equations $\Delta_{j_1}(x_p + \varepsilon z_p) = \Delta_j(x_p + \varepsilon z_p)$, i.e., from the expressions

$$\varepsilon = \frac{\Delta_j(x_p) - \Delta_{j_1}(x_p)}{\Delta_{j_1}(z_p) - \Delta_j(z_p)} \qquad (j \neq j_1, \dots, j_{p_1})$$

and the least positive $\varepsilon = \varepsilon_p''$ from the equations $\delta_k(x_p + \varepsilon z_p) = l_k$, i.e., from the expressions

$$\varepsilon = \frac{l_k - \delta_k(x_p)}{\delta_k(z_p)} \qquad (k \neq k_1, \dots, k_{p_2}).$$

Set $\varepsilon_p = \min\{\varepsilon_p', \varepsilon_p''\}$ and $x_{p+1} = x_p + \varepsilon_p z_p$.

3. Continuing the process, we arrive at a stationary point x_q, i.e., such that, for example,

$$\Delta_{j_1}(x_q) = \dots = \Delta_{j_{q_1}}(x_q) < \Delta_j(x_q) \qquad (j \neq j_1, \dots, j_{q_1});$$

$$\delta_{k_1}(x_q) = l_{k_1}, \dots, \delta_{k_{q_2}}(x_q) = l_{k_{q_2}}; \quad \delta_k(x_q) > l_k \ (k \neq k_1, \dots, k_{q_2}); \quad q_1 + q_2 = q,$$

and there is no direction z_q such that, in moving along it (i.e., as ε increases), all the distances $\Delta_{j_1}(x_q + \varepsilon z_q), \dots, \Delta_{j_{q_1}}(x_q + \varepsilon z_q)$ would increase while remaining equal to one another, and the point $x_q + \varepsilon z_q$ would remain in the same boundary planes of Ω as the point x_q, i.e., $\delta_{k_1}(x_q + \varepsilon z_q) = l_{k_1}, \dots, \delta_{k_{q_2}}(x_q + \varepsilon z_q) = l_{k_{q_2}}$. In other words, we encounter the fact that from equations $2')$ and $3')$ it will follow that $\sum_{i=1}^{n} b_{j_1 i}\zeta_i = 0$, in particular, that the system $2')$–$3')$ has only the trivial solution.

Consider the system

$$\Delta_{j_1}(x) = 0, \dots, \Delta_{j_{q_1}}(x) = 0; \quad \delta_{k_1}(x) = l_{k_1}, \dots, \delta_{k_{q_2}}(x) = l_{k_{q_2}}, \tag{5}$$

and let the rank of the matrix of its coefficients be equal to r_0 $(r_0 \leq n)$. This means that among (5) there are subsystems of r_0 planes which intersect, i.e., have a common linear manifold—an edge of dimension $n - r_0$ (for $r_0 = n$ the edge degenerates into a point). All such edges are mutually parallel.

From the point x_q draw an r_0-dimensional plane perpendicular to these edges. We shall call the points of intersection vertices. Let us find the characteristics of these vertices, where by the characteristic of a vertex x' we shall mean the sum of the number of planes from (5) passing through x', the number of planes $\Delta_j(x) = 0$ from (5) separating x' from the point x_q, and the number of planes $\delta_k(x) = l_k$ from (5) for which $\delta_k(x') < l_k$.* If the characteristic of every vertex is less than q, then the point x_q is optimal. In this case x_q belongs to some prism (in particular, to some simplex) bounded by the planes from (5). The process terminates here.

If, however, the characteristic of some vertex x' is equal to q, then we move from x_q along the straight line $\overline{x'_{qx}}$ in the direction away from x'. In doing so we shall not leave Ω, and the distances to all the planes $\Delta_{j_\nu}(x) = 0$ $(\nu = 1, \ldots, q_1)$ increase; moreover, the distances to those of them that pass through the vertex x' remain equal to one another, while the distances to the separating planes increase more rapidly. One should move along the straight line $\overline{x'_{qx}}$ until meeting a new stationary point. After a finite number of steps we shall arrive at the optimal point x^*, and the process will be completed.

*

If all the planes (5) pass through x_q, then one must shift all the planes $\Delta_{j_1}(x) = 0, \ldots, \Delta_{j_{q_1}}(x) = 0$ parallel to themselves by ε so that the distances from x_q to them become positive. Then we find the vertices for the new position of the planes and move from the vertex x', with characteristic q, returning the planes to their original position and preserving only the direction $x'x_p$.

4. Along the way we have established the following optimality criterion: in order that the point x^* be optimal, it is necessary and sufficient that it be stationary and that it belong to some r-dimensional prism bounded by r_1 planes from (1), from which x^* is equally and maximally distant, and by r_2 planes $\delta_k(x) = l_k$, to which it belongs $(r_1 + r_2 = r + 1 \leq n + 1)$. The optimal point for the system of planes and inequalities (1)–(2) is also an optimal point for this subsystem of $r_1 + r_2$ planes and inequalities, and $\max_{x\in\Omega} \min_j \Delta_j(x)$ is the same in both problems.

Let us note that the coefficients $\alpha_{k_1}, \ldots, \alpha_{k_{r_2}},\ \beta_{j_1}, \ldots, \beta_{j_{r_1}}$ of the linear dependence

$$\alpha_{k_1}\delta_{k_1}(x) + \ldots + \alpha_{k_{r_2}}\delta_{k_{r_2}}(x) + \beta_{j_1}\Delta_{j_1}(x) + \ldots + \beta_{j_{r_1}}\Delta_{j_{r_1}}(x) \equiv 0$$

of the faces of the prism are positive. These multipliers are the so-called resolving multipliers of L. V. Kantorovich ([1]) (for the remaining planes of system (1) and the boundary of Ω, the resolving multipliers are equal to zero).

5. **Remarks.**

1) The case in which among the conditions (2) there are equalities $\delta_k(x) = l_k$ is reduced to the case considered by replacing each such equality by the inequalities $\delta_k(x) \geq l_k,\ -\delta_k(x) \geq -l_k$.

2) One may also use the following criterion for optimality of a stationary point x_q: translate in parallel the polyhedral cone bounded by the planes $\Delta_j(x) = 0$ from (5), placing its vertex at x_q. If x_q is the only point of Ω belonging to the cone, then it is optimal. Otherwise it is not optimal.

3) It is not difficult to see how to modify the algorithm for finding

$$\min_{x\in\Omega}\max_{1\le j\le s}\Delta_j(x),$$

and also

$$\min_{x\in\Omega}\max_{1\le j\le s}|\Delta_j(x)|$$

(the latter problem is essentially the discrete case of V. A. Markov' s generalized problem ([5])). The algorithm is also suitable for finding in Ω the point most (least) distant from a given system of planes, not necessarily passing through one point.

4) The algorithm presented is also suitable for solving a system of linear inequalities

$$\delta_k(x)\equiv\sum_{i=1}^{n}a_{ik}\xi_i\ge l_k\quad(k=1,\dots,m+n).$$

We begin the process with an arbitrary point x'; only the role of the planes (1) is played by those planes $\delta_k(x)=l_k$ for which $\delta_k(x')<l_k$, and the role of the inequalities (2) by the remaining inequalities.

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Note: Figure translations are in progress. See original paper for figures.