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Abstract

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MATHEMATICS

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THE FIRST BOUNDARY-VALUE PROBLEM FOR QUASILINEAR ELLIPTIC EQUATIONS

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1. Let Ω be a domain in the plane x, y, bounded by a closed convex curve Γ. With respect to Γ we shall assume that it is given parametrically by functions $x = x(s)$, $y = y(s)$ that are three times continuously differentiable (s is the arc length on Γ), and that its curvature at all points is not less than $\varkappa_0 = \text{const} > 0$.

In the domain $\Omega+\Gamma$ consider the first boundary-value problem for the quasilinear elliptic equation

$$A(x, y, z, p, q)r + 2B(x, y, z, p, q)s + C(x, y, z, p, q)t = D(x, y, z, p, q) \qquad (1)$$

with zero boundary condition.

We shall assume that the functions A, B, C, D are continuously differentiable in all the variables x, y, z, p, q, and that their first derivatives satisfy a Hölder condition with exponent $0 < \beta \leq 1$ with respect to $(x, y) \in \Omega + \Gamma$ and with respect to z, p, q for all finite values of these variables.

Let

$$|D(x, y, 0, p, q)| \leq \varphi(x, y)R\left(\sqrt{p^2 + q^2}\right),$$

where $\varphi(x, y) \geq 0$ and $R\left(\sqrt{p^2 + q^2}\right) > 0$ are continuous functions of their variables, the first in the domain $\Omega + \Gamma$, the second in the plane p, q.

Using the function $\varphi(x, y)$, we construct several auxiliary functions. Let M be an arbitrary point of the curve Γ. Denote by K_M the circle of smallest radius that is tangent to Γ at the point M and contains the domain $\Omega+\Gamma$ in its interior. Let O_M be the center of this circle and r_M its radius. On the interval $[0, r_M]$ we construct the function

$$f_M(\rho) = \max \varphi(x, y),$$

where the greatest value of the function $\varphi(x, y)$ is taken on the arc of the circle with center at O_M and radius $0 \le \rho \le r_M$, lying inside $\Omega + \Gamma$. Let M still be an arbitrary point on Γ, and let t_M be the tangent to Γ at the point M. Project the domain $\Omega + \Gamma$ onto the line l_M perpendicular to t_M. Introduce in the plane x, y a new Cartesian coordinate system u, v so that the lines l_M, t_M are respectively the u- and v-axes. Let $u_M^{(1)}$ and $u_M^{(2)}$ ($u_M^{(1)} < u_M^{(2)}$) be the endpoints of the segment on the u-axis* which is the projection of the domain $\Omega + \Gamma$. Put

$$\mu_M(u) = \max_v \varphi(u, v),$$

* One of the numbers $u_M^{(1)}, u_M^{(2)}$ is equal to zero.

where the greatest value of the function $\varphi(u, v)$ is taken over all points of the domain $\Omega + \Gamma$ with a common projection onto the straight line u. The constructed function is defined on the interval $[u_M^{(1)}, u_M^{(2)}]$.

The following theorems on the solvability of the Dirichlet problem for equation (1) hold.

Theorem 1. *Suppose that in the domain $\Omega + \Gamma$ there is given the equation*

$$A(x, y, p, q)r + 2B(x, y, p, q)s + C(x, y, p, q)t = D(x, y, z, p, q), \qquad (2)$$

whose coefficients satisfy the conditions considered above, as well as the following requirements:

a) *The inequality holds*

$$A(x, y, p, q)\xi^2 + 2B(x, y, p, q)\xi\eta + C(x, y, p, q)\eta^2 \ge$$

$$\ge \alpha\left(\sqrt{p^2 + q^2}\right)(\xi^2 + \eta^2),$$

where $\alpha\left(\sqrt{p^2 + q^2}\right)$ is a continuous function on the p, q-plane, strictly positive for all finite values of p, q.

b) *For all $(x, y) \in \Omega + \Gamma$ and arbitrary z, p, q, the inequality*

$$D_z(x, y, z, p, q) \ge \text{const} > 0$$

is valid.

c) *For fixed x, y, z, the ratios $|D_x/D_z|$, $|D_y/D_z|$, $|C_x/C|$, $|A_y/A|$, $|D/D_z|$ are uniformly bounded as functions of z.*

d) *One of the two relations is fulfilled*

$$\sup_{M\in\Gamma}\left\{\pi\int_0^{r_M} f_M(r)\,r\,dr\right\} < 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\alpha^2\left(\sqrt{p^2+q^2}\right)}{R^2\left(\sqrt{p^2+q^2}\right)}\,dp\,dq \tag{3}$$

or

$$\sup_{M\in\Gamma}\int_{u_M^{(1)}}^{u_M^{(2)}}\mu_M(u)\,du < \min\left\{\int_{-\infty}^{0}\frac{\alpha(\tau)}{R(\tau)}\,d\tau,\ \int_0^{+\infty}\frac{\alpha(\tau)}{R(\tau)}\,d\tau\right\}. \tag{4}$$

Then the first boundary value problem for equation (2) with zero boundary condition is uniquely solvable in the class of functions whose third derivatives satisfy in the domain $\Omega+\Gamma$ a Hölder condition with exponent $0<\beta'<\beta$.

Theorem 2. *Suppose that all conditions of Theorem 1 are fulfilled, except condition b), which we replace by the following:*

b′) *Suppose that for all $x, y \in \Omega+\Gamma$ and arbitrary z, p, q, the inequality*

$$D_z(x,y,z,p,q) \geq 0$$

is valid, the functions A, B, C, D, for fixed x, y and z, behave as polynomials in the variables p, q as $p^2+q^2 \to +\infty$, and the function $D(x,y,z,p,q)$ is represented in the form

$$D(x,y,z,p,q) = D_1(x,y,z,p,q) + D_2(x,y,z,p,q),$$

where

$$D_1(x,y,z,p,q) \geq 0$$

for all $(x,y)\in\Omega+\Gamma,\ z,p,q$;

$$\frac{D_2}{Ap^2+2Bpq+Cq^2} \leq k = \text{const} < +\infty$$

for $(x,y)\in\Omega+\Gamma,\ p^2+q^2>1$, and arbitrary z.

Then the conclusion of Theorem 1 holds.

Theorem 3. Suppose that in the domain $\Omega+\Gamma$ the equation

$$A(x,y,z,p,q)r + 2B(x,y,z,p,q)s + C(x,y,z,p,q)t = D(x,y,z,p,q), \tag{5}$$

is given, whose coefficients satisfy the following conditions:

a) The functions A, B, C, D are continuously differentiable with respect to all variables, and their first derivatives satisfy the Hölder condition with exponent $0<\beta\leqslant 1$ in $(x,y)\in\Omega+\Gamma$ and for all finite z, p, q.

Further, for all admissible values of x, y, z, p, q, the quadratic form

$$A_z(x,y,z,p,q)\xi^2 + 2B_z(x,y,z,p,q)\xi\eta + C_z(x,y,z,p,q)\eta^2 \leqslant -\lambda_0(\xi^2+\eta^2)$$

$$(\lambda_0 = \text{const} > 0),$$

and, finally,

$$A(x,y,0,p,q)\xi^2 + 2B(x,y,0,p,q)\xi\eta + C(x,y,0,p,q)\eta^2 \geqslant$$

$$\geqslant \alpha\left(\sqrt{p^2+q^2}\right)(\xi^2+\eta^2),$$

where $\alpha\left(\sqrt{p^2+q^2}\right)$ is a continuous function on the p,q-plane, strictly positive for all finite values of p, q.

b) For all admissible values of x, y, z, p, q we have

$$D_z(x,y,z,p,q) \geqslant 0, \qquad D(x,y,z,p,q) \geqslant \text{const} > 0.$$

c) Condition c) of Theorem 1 is fulfilled and, moreover, A_y, A_z, A; C_x, C_z, C; D_x, D_y, D_z, D have the same orders of growth in p, q.

d) Condition d) of Theorem 1 is fulfilled.

Then the first boundary-value problem with zero boundary condition for equation (5) has at least one solution in the class of functions whose third derivatives satisfy the Hölder condition in the domain $\Omega + \Gamma$ with exponent $\beta' < \beta$.

Let us note that if the function $\mu_M(u)$ is constructed so that its graph is symmetric with respect to the ordinate passing through the midpoint of the interval $[u_M^{(1)}, u_M^{(2)}]$ (for any point $M \in \Gamma$), then condition (4) may be replaced by the following:

$$\sup_{M\in\Gamma} \int_{u_M^{(1)}}^{u_M^{(2)}} \mu_M(u)\,du < \int_{-\infty}^{+\infty} \frac{\alpha(\tau)}{R(\tau)}\,d\tau.$$

2. Theorems 1 and 2 represent a further development of S. N. Bernstein' s results on quasilinear elliptic equations of the form (2) (see ([1])). S. N. Bernstein considers equations for which the coefficients A, B, C, D have coordinated orders of growth in p, q as $p^2 + q^2 \to +\infty$. This coordination is expressed by the condition that for $p^2 + q^2 > 1$

$$\frac{|D|}{Ap^2 + 2Bpq + Cq^2} \leqslant k = \text{const} < +\infty. \qquad (6)$$

The first boundary-value problem for the equation

$$(1 + q^2)r - 2pqs + (1 + p^2)t = (1 + p^2 + q^2)^{3/2},$$

which does not obey condition (6), is solvable, generally speaking, only in sufficiently small domains.

The integral conditions d) in Theorems 1 and 2 give sufficient conditions for the solvability of the first boundary-value problem for quasilinear elliptic equations, if the orders of growth of the coefficients with respect to the first derivatives do not obey S. N. Bernstein' s coordination condition. Let us note that in a number of cases these conditions are very close to necessary ones; for example, if

the domain Ω is a circle of radius 1, and equation (2) has the form

$$(1 + q^2)r - 2pqs + (1 + p^2)t = (1 + p^2 + q^2)^{3/2}.$$

Geometrically, this concerns the construction of a surface with mean curvature equal to one. The solution of the problem under consideration will be a hemisphere of radius one.

Theorem 3 is an analogue of the results of J. Schauder ([2]), L. Nirenberg ([3]), and O. A. Ladyzhenskaya for equations

$$A(x, y, z, p, q)r + 2B(x, y, z, p, q)s + C(x, y, z, p, q)t = 0$$

in the general case of nonhomogeneous quasilinear elliptic equations.

3. Let us briefly outline the proof of the theorems stated. From the results of S. N. Bernstein ([1]), J. Schauder ([2]), and L. Nirenberg ([3]), it follows that, for this purpose, it suffices to prove the possibility of obtaining a priori estimates for the proposed solution in the space C^1. We first obtain an a priori estimate for the modulus of the solution in $\Omega + \Gamma$ and for its first derivatives on the boundary. To this end, a convex shell is stretched over the solution of the equation; it consists of two convex caps, with their convexities turned in opposite directions and joined to each other along the curve Γ. Conditions b) and c) in Theorems 1 and 2, and conditions a), b), and c) in Theorem 3, make it possible to give an a priori estimate for the slopes of the supporting planes on the boundary of both caps; from this the a priori estimates required for us for the modulus and the first derivatives on the boundary follow easily. A priori estimates for the

moduli of the first derivatives of the solution in the interior are obtained by means of S. N. Bernstein' s method of auxiliary functions.

After this, the completion of the proof of Theorems 1, 2, and 3 is obtained by reduction to Schauder' s topological principle, using the results of Nirenberg and Schauder on linear elliptic equations (this scheme is described in detail in ([3])).

In conclusion we note that the results obtained above are naturally generalized to the case of a sufficiently smooth nonhomogeneous boundary condition; for lack of space we shall not dwell on them.

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REFERENCES

[1] S. N. Bernstein, UMN, 8, 75 (1941).
[2] J. Schauder, Math. Zs., 34, No. 4 (1933).
[3] L. Nirenberg, Comm. Pure and Appl. Math., 6, No. 3 (1953).
[4] O. A. Ladyzhenskaya, DAN, 120, No. 5 (1958).

Note: Figure translations are in progress. See original paper for figures.