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1960

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Abstract

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Distribution of Integer Points on Certain Norm Surfaces

(Presented by Academician I. M. Vinogradov on 19 IV 1960)

§ 1. Distribution of integer points on certain ellipses and hyperbolas.

Let $\omega = \omega(a, b)$ be the angle between the rays issuing from the origin and making angles a and b with the abscissa axis, where $0 \le a < b \le 2\pi$; $\omega = b - a$; $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$; $\delta = 1$, if $-d \equiv 2, 3 \pmod 4$; $\delta = 4$, if $-d \equiv 1 \pmod 4$. Let m be an odd natural number,

$$m = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s}$$

its canonical decomposition into prime factors; r the number of all integer points lying on the ellipse

$$x^2 + dy^2 = \delta m; \tag{1}$$

T the number of integer points on the ellipse (1) that lie inside the angle of aperture ω.

Theorem 1. If

$$\left(\frac{-d}{p_t}\right) = +1$$

for $t = 1, 2, \dots, s$ (where $\left(\frac{-d}{p_t}\right)$ is the Legendre symbol) and

$$\Delta = \sqrt{\frac{\ln(p_1 \cdots p_s)}{\ln r}} \to 0 \quad \text{as } m \to \infty,$$

then

$$T = \frac{r}{2\pi}\left[\operatorname{arc\,tg}(\sqrt{d}\ \operatorname{tg} b) - \operatorname{arc\,tg}(\sqrt{d}\ \operatorname{tg} a)\right] + O(r\Delta).$$

Proof. We note the main stages of the proof for the simplest case $d = 1$. The proof for other d is analogous.

Let

$$m = x_j^2 + y_j^2, \qquad x_j + iy_j = \sqrt{m}\,\exp(2\pi i\alpha_j), \qquad 0 \le \alpha_j < 1,$$

where x_j, y_j are integers, $j = 1, 2, \dots, r$. We are interested in those solutions for which

$$0 < A \le \alpha_j \le B < 1, \qquad A < B, \qquad b - a = 2\pi(B - A).$$

Estimate the sum

$$S_m^{(k)} = \sum_{j=1}^{r} \exp(2\pi i k\alpha_j),$$

where k is an integer, $k \ne 0$.

All integers $x_j + iy_j$ of the field $R(i)$ (R is the field of rational numbers) with norm m are represented in the form

$$x_j + iy_j = (i)^{\gamma} \prod_{t=1}^{s} c_t^{\gamma_t} {c'}_t^{\beta_t - \gamma_t},$$

where c_t, c_t' are prime numbers of the field $R(i)$, with $c_t c_t' = p_t$, $\gamma = 0, 1, 2, 3$, $\gamma_t = 0, 1, \dots, \beta_t$. Taking into account that $c_t^k \ne {c'}_t^k$ for any $k \ne 0$, we obtain

$$\left|S_m^{(k)}\right| \le \min\left(r,\ 4(p_1 p_2 \cdots p_s)^{k/2}\right).$$

Further, applying Lemma 12 from [1], p. 260, we obtain the proof of the theorem.

Let d be a natural square-free number, $d > 1$; $R(\sqrt{d})$ a one-class field (R is the field of rational numbers); ε a unit of norm 1 such that every unit of norm 1 is obtained in the form $\pm\varepsilon^n$, where n is some integer. We may assume that $\varepsilon > 1$. Let φ be the angle between the ray

$$y = \frac{\varepsilon^2 - 1}{(\varepsilon^2 + 1)\sqrt{d}}\,x, \qquad x \geqslant 0,$$

and the axis of abscissas; let $\omega = \omega(a, b)$ be the angle between rays issuing from the origin and making angles a and b with the axis of abscissas, where

$$\omega = b - a, \qquad 0 \leqslant a < b < \operatorname{arc\,tg} \frac{\varepsilon^2 - 1}{(\varepsilon^2 + 1)\sqrt{d}}.$$

Let m be odd,

$$m = p_1^{\beta_1} p_2^{\beta_2} \cdots p_s^{\beta_s}, \qquad \left(\frac{d}{p_t}\right) = +1 \quad \text{for } t = 1, 2, \ldots, s;$$

r is the number of integral points on the hyperbola

$$x^2 - dy^2 = \delta m, \tag{2}$$

lying inside the angle φ; T is the number of integral points on the hyperbola (2) lying inside the angle ω, where $\delta = 1$, if $d \equiv 2, 3 \pmod 4$, and $\delta = 4$, if $d \equiv 1 \pmod 4$. Put

$$u = \frac{(1 - \sqrt{d}\ \mathrm{tg}\, a)(1 + \sqrt{d}\ \mathrm{tg}\, b)}{(1 - \sqrt{d}\ \mathrm{tg}\, b)(1 + \sqrt{d}\ \mathrm{tg}\, a)}, \qquad \Delta = \sqrt{\frac{\ln(\varepsilon^{3s} p_1 \cdots p_s)}{\ln\left[r(\varepsilon \ln \varepsilon)^{-s}\right]}}.$$

Theorem 2. If $\Delta \to 0$ as $m \to \infty$, then

$$T = \frac{r \ln u}{2 \ln \varepsilon} + O(r\Delta).$$

Proof. We indicate the main stages of the proof. The hyperbola (2) can be written in the form $\xi\eta = m$. Under the logarithmic mapping a point (ξ, η) with $\xi > 0$, $\eta > 0$ passes to the point $(v, w) = (\ln \xi, \ln \eta)$ of the plane (v, w); the hyperbola $\xi\eta = m$ to the straight line $v + w = \ln m$; the angle $\omega(a_1, b_1)$, where $a_1 \neq 0$, $b_1 \neq \pi/2$, to the strip bounded by the straight lines $w = v + \ln \mathrm{tg}\, a_1$ and $w = v + \ln \mathrm{tg}\, b_1$; the unit ε is represented in the form of the vector $l(\varepsilon) = (\ln \varepsilon, \ln \varepsilon')$, where ε' is conjugate to ε. The angle φ passes to the strip bounded by the straight lines $w = v$ and $w = v - 2 \ln \varepsilon$.

Let $p_t = c_t c_t'$, where c_t, c_t' are prime numbers of the field $R(\sqrt{d})$,

$$\mu = \left| \prod_{t=1}^{s} c_t^{\gamma_t} c_{t'}^{\beta_t - \gamma_t} \right|, \qquad \gamma_t = 0, 1, 2, \ldots, \beta_t.$$

It is clear that $N(\mu) = \mu\mu' = m$. The number of such μ will be r.

Estimate the sum

$$S_m^{(k)} = \sum_{\mu} \exp\left\{2\pi i k \frac{\ln(\mu/\sqrt{m})}{\ln \varepsilon}\right\},$$

where k is an integer, $k \neq 0$, and μ runs through the indicated r numbers.

Taking into account that $\ln |c_t/c_t'| / \ln \varepsilon$ is an irrational number, we obtain

$$\left|S_m^{(k)}\right| \leqslant \min\left(r,\ \prod_{t=1}^{s} \frac{1}{2\,(\ln|c_t/c_t'|^k/\ln\varepsilon)}\right),$$

where ( ) denotes the distance to the nearest integer. Estimating this distance from below, we obtain

$$\left|S_m^{(k)}\right| \leqslant \min\left(r,\ (\varepsilon\ln\varepsilon)^s(\varepsilon^{3s}p_1\cdots p_s)^k\right).$$

Further, applying Lemma 12 of paper ([1]), p. 260, and passing to the variables x, y, we obtain Theorem 2.

Remark 1. For fixed s and prime numbers $p_1, p_2, \dots, \dots, p_s$, the application of A. O. Gelfond' s theorem ([2]) for estimating (   ) from below gives a better remainder term in Theorem 2.

Remark 2. Using the methods of proof of Theorems 1 and 2, one can obtain theorems on the distribution of integer points on a system of nonequivalent ellipses and hyperbolas, but in view of the cumbersomeness of these theorems we do not give them here.

§ 2. **Distribution of integer points on certain surfaces of the third order.** Let $R(\theta)$ be a one-class field of degree 3 over the field of rational numbers, where θ is a nonreal number; $\omega_1, \omega_2, \omega_3$ is a basis of the integers of the field $R(\theta)$. In three-dimensional space (ξ_1, ξ_2, ξ_3) consider the lattice Γ:

$$\xi_1 = x_1 \operatorname{Re}\omega_1 + x_2 \operatorname{Re}\omega_2 + x_3 \operatorname{Re}\omega_3,$$

$$\xi_2 = x_1 \operatorname{Im}\omega_1 + x_2 \operatorname{Im}\omega_2 + x_3 \operatorname{Im}\omega_3,$$

$$\xi_3 = x_1\omega_1'' + x_2\omega_2'' + x_3\omega_3'',$$

where ω_i'' are real conjugates of ω_i; x_1, x_2, x_3 are rational integers.

Consider the surface

$$(\xi_1^2 + \xi_2^2)|\xi_3| = m, \tag{3}$$

where m is a natural number. It is known that in the field $R(\theta)$ there exist infinitely many prime numbers c such that $|cc'c''| = p$, where p is a rational prime number; c' is the complex conjugate of c; c'' is the real conjugate of c. We assume that m consists exclusively of the product of powers of such prime numbers: $m = p_1^{\beta_1}p_2^{\beta_2}\cdots p_s^{\beta_s}$. On the surface (3) we distinguish the region D_m: $\sqrt[3]{m} \le |\xi_3| \le \sqrt[3]{m}\,|\varepsilon|^2$, where ε is the fundamental unit of the field $R(\theta)$. Let r

denote the number of points of the lattice Γ that fall in D_m. We distinguish a part of the region D_m: $\sqrt[3]{m} \le \alpha\sqrt[3]{m} \le |\xi_3| \le \beta\sqrt[3]{m} \le \sqrt[3]{m}\,|\varepsilon|^2$. By T we denote the number of points of the lattice Γ falling in this part of D_m. It is not difficult to compute that

$$r = 2\prod_{t=1}^{s} \frac{(\beta_t+1)(\beta_t+2)}{2}.$$

Denote

$$\Delta = \sqrt{\frac{\ln\left(|\varepsilon|^{108s}(p_1\dots p_s)^3\right)}{\ln\left(r2^{-6s}|\varepsilon|^{-12s}\ln^{-3s}|\varepsilon|\right)}}.$$

Theorem 3. If $\Delta \to 0$ as $m \to \infty$, then

$$T = \frac{\ln\beta - \ln\alpha}{2\ln|\varepsilon|}\,r + O(r\Delta).$$

The proof of this theorem is analogous to the proof of Theorem 2.

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Received
18 IV 1960

CITED LITERATURE

[1] I. M. Vinogradov, *Selected Works*, 1952.
[2] A. O. Gelfond, *Transcendental and Algebraic Numbers*, 1952, p. 217.

Note: Figure translations are in progress. See original paper for figures.