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V. D. BONDAR’

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Abstract

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V. D. BONDAR'

SOME EXACT SOLUTIONS OF THE COMPATIBILITY EQUATIONS FOR THE COMPONENTS OF THE STRAIN TENSOR UNDER SIMPLE LOADING

(Presented by Academician L. I. Sedov, 9 XI 1959)

In the work of L. I. Sedov ([1]) it is shown that simple loading corresponds to a deformation of a certain special type, and compatibility equations were obtained which must be satisfied by the components of the finite-strain tensor ε_{ij} under simple loading. Below we shall consider the compatibility equations in a more detailed form and indicate two exact solutions of these equations.

In ([1]) the compatibility equations for the quantities ε_{ij} are given in the form

$$\frac{\partial G_{\nu\mu i}}{\partial \xi^j} - \frac{\partial G_{\nu\mu j}}{\partial \xi^i} = 0, \tag{1}$$

$$g^{*\alpha\omega}\left(G_{\omega\mu j}G_{\alpha\nu i} - G_{\omega\mu i}G_{\alpha\nu j}\right) = 0, \tag{2}$$

where

$$G_{\nu\alpha j} = \frac{\partial \varepsilon_{\alpha\nu}}{\partial \xi^j} + \frac{\partial \varepsilon_{j\nu}}{\partial \xi^\alpha} - \frac{\partial \varepsilon_{\alpha j}}{\partial \xi^\nu}, \qquad \left\|g^{*\alpha\omega}\right\| = \left\|g^0_{\alpha\omega} + 2k\varepsilon_{\alpha\omega}\right\|^{-1}; \tag{3}$$

$g^0_{\alpha\omega}$ is the unit tensor, and the coefficient k may assume arbitrary values in the interval $(0, 1)$. In these and subsequent formulas, unless specially stated, summation is performed over repeated indices α and ω. The independent equations correspond to the following systems of indices:

$$ij\mu\nu = 1212,\ 1313,\ 2323,\ 1213,\ 2123,\ 3132. \tag{4}$$

The matrix relation (3) may be represented in the form

$$\left\|g^{*\alpha\omega}\right\| = \frac{1}{\Delta}\left\|g^0_{\alpha\omega}\right\| + \frac{2k}{\Delta}\left(I^0_1\left\|g^0_{\alpha\omega}\right\| - \left\|\varepsilon_{\alpha\omega}\right\|\right) + \frac{4I^0_3 k^2}{\Delta}\left\|\varepsilon_{\alpha\omega}\right\|^{-1},$$

where

$$\Delta = \left|g^0_{\alpha\omega} + 2k\varepsilon_{\alpha\omega}\right| = 1 + 2I^0_1 k + 4I^0_2 k^2 + 8I^0_3 k^3;$$

$I^0_1,\ I^0_2,\ I^0_3$ are the invariants of the finite-strain tensor referred to the basis of the initial states. With the use of this relation the system of equations (2), in view of the arbitrariness of the coefficient k, splits into three systems of equations

$$B_{\alpha\alpha ij\mu\nu} = 0, \qquad \varepsilon_{\alpha\omega} B_{\omega\alpha ij\mu\nu} = 0, \qquad \varepsilon'_{\alpha\omega} B_{\omega\alpha ij\mu\nu} = 0.$$

Here the notations introduced are

$$B_{\omega\alpha ij\mu\nu} = G_{\omega\mu j} G_{\alpha\nu i} - G_{\omega\mu i} G_{\alpha\nu j}, \qquad \|\varepsilon'_{\alpha\omega}\| = \|\varepsilon_{\alpha\omega}\|^{-1}.$$

Thus, under a deformation corresponding to simple loading, the components of the finite-strain tensor must satisfy the following 24 compatibility equations:

$$\frac{\partial G_{\nu\mu i}}{\partial \xi^j} - \frac{\partial G_{\nu\mu j}}{\partial \xi^i} = 0, \tag{5}$$

$$B_{\alpha\alpha ij\mu\nu} = 0, \qquad \varepsilon_{\alpha\omega} B_{\omega\alpha ij\mu\nu} = 0, \qquad \varepsilon'_{\alpha\omega} B_{\omega\alpha ij\mu\nu} = 0.$$

In the general case these equations are nonlinear partial differential equations of the second order.

As indicated in (1), a homogeneous (affine) deformation $\varepsilon_{ij} = \text{const}$ admits simple loading. Other solutions of system (5) can be found from the following considerations. Setting

$$G_{\nu\mu i} = G_{\nu i\mu} = \frac{\partial \psi_{\nu\mu}}{\partial \xi^i},$$

where $\psi_{\nu\mu}$ are certain functions of the coordinates ξ^1, ξ^2, ξ^3, we satisfy the first 6 equations of system (5); in this case, for the components of the finite-strain tensor and the quantities $B_{\omega\alpha ij\mu\nu}$ we have the formulas

$$\varepsilon_{ij} = \frac{1}{2}(\psi_{ij} + \psi_{ji}), \qquad B_{\omega\alpha ij\mu\nu} = \frac{D(\psi_{\omega\mu}, \psi_{\alpha\nu})}{D(\xi^j, \xi^i)}. \tag{6}$$

It is clear that the remaining equations of system (5) are satisfied if we require that the functions ψ_{ij} satisfy the equations

$$B_{\omega\alpha ij\mu\nu} = 0, \tag{7}$$

where $\alpha, \omega = 1, 2, 3$. This means that all the functions ψ_{ij} are dependent on one another and, consequently, may, generally speaking, be expressed through one of them. By virtue of equality (6), all components ε_{ij} will be arbitrary functions of one component, for example ε_{11}:

$$\varepsilon_{ij} = f_{ij}(\varepsilon_{11}), \qquad i, j = 1, 2, 3. \tag{8}$$

Instead of conditions (7), one may subject the functions ψ_{ij} to the equations

$$B_{\omega\alpha ij\mu\nu} = 0 \quad \text{for } \alpha = \omega;$$

$$B_{\omega\alpha ij\mu\nu} = -B^*_{\alpha\omega ij\mu\nu} \quad \text{for } \alpha \neq \omega, \qquad \alpha, \omega = 1, 2, 3, \tag{9}$$

and then the compatibility equations will be satisfied. From equalities (9), along with the preceding solution, there follows a solution in which, among the functions ψ_{ij}, three with distinct first indices are arbitrary, while the functions with identical first indices depend on one another according to a linear law. Namely, choosing as arbitrary functions $\psi_{11}, \psi_{22}, \psi_{33}$, we shall have:

$$\psi_{12} = a\psi_{11}, \quad \psi_{13} = b\psi_{11}; \quad \psi_{21} = \frac{1}{a}\psi_{22}, \quad \psi_{23} = \frac{b}{a}\psi_{22}; \quad \psi_{31} = \frac{1}{b}\psi_{33}, \quad \psi_{32} = \frac{a}{b}\psi_{33},$$

where a and b are arbitrary constants. This corresponds to the following values of the components of the finite-strain tensor: the components $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}$ are arbitrary, and

$$\varepsilon_{12} = \frac{1}{2}\left(a\varepsilon_{11} + \frac{1}{a}\varepsilon_{22}\right), \quad \varepsilon_{13} = \frac{1}{2}\left(b\varepsilon_{11} + \frac{1}{b}\varepsilon_{33}\right), \quad \varepsilon_{23} = \frac{1}{2}\left(\frac{b}{a}\varepsilon_{22} + \frac{a}{b}\varepsilon_{33}\right). \tag{10}$$

Steklov Mathematical Institute
Academy of Sciences of the USSR

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