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Abstract

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ON THE METHOD OF NETS FOR THE THIRD BOUNDARY-VALUE PROBLEM

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In the present paper the error arising in the application of difference schemes considered in (4) is estimated; for this purpose a difference Green' s function is constructed and its properties on the boundary are investigated; at the end of the paper some difference schemes approximating the third boundary condition are given.

Let the function $U(x_1, x_2)$ be a harmonic function in the domain $\Omega(x_1, x_2)$, whose boundary S consists of a finite number of twice continuously differentiable curves without points of return, and at each point of it one can touch the vertex of a sector of radius r_0 and angle $\pi\theta$, $|\theta| < 1$, lying entirely outside Ω. For simplicity we shall assume that the domain Ω is simply connected. In addition, let $U \in H(1+\rho, A, r)$ (3), and on the boundary S let the condition be satisfied

$$\frac{\partial U}{\partial n} + aU = \varphi, \tag{1}$$

where n is the exterior normal; a and φ are piecewise smooth functions of the parameter s (the length of the contour), measured from the point s_0 in the direction of positive traversal, and $a \geq c_0 > 0$.

We briefly reproduce the constructions of papers (4-6). Construct, with step h, the net Ω_h; denote the boundary points of this set by S_h, denote the set of centers of squares of the net domain Ω_h by Ω'_h, and the boundary points of this set by S'_h; let this be the set of points s'_k, $k = 0, 1, \ldots, N$, renumbered in the direction of positive traversal of the contour; we put them in correspondence with points $s_k \in S$ so that $\Delta s_k = O(h)$ (1). In what follows, by

$$\frac{\Delta}{\Delta s}, \quad \frac{\Delta}{\Delta s'}$$

we shall denote differences divided by h, respectively along S_h and S'_h. Let $V(x_1, x_2)$ be the function conjugate to U; Ω_1 a strictly interior subdomain of the domain Ω; c_i positive constants independent of h; $\bar{a}_k = \frac{\Delta s_k}{h} a_k$; for simplicity the indices k, h are sometimes omitted.

On the net Ω_h construct an h-harmonic function u, satisfying on the set S_h the condition (4-6)

$$\frac{\Delta u}{\Delta n} + \bar{a} u = \bar{\varphi}. \tag{2}$$

By a simple transfer to the discrete case of methods for estimating harmonic functions satisfying condition (1) (7), one can obtain that in Ω_h

$$|u| \leq \frac{\max|\varphi|}{\min a}. \tag{3}$$

Difference schemes for which the function u satisfies inequality (3) we shall conditionally call **positive**.

I. To estimate the error, we construct the difference Green’ s function $G(x_1, x_2, y_1, y_2)$ of our problem:

$$\Delta_h G = \begin{cases} -h^{-2}, & \text{for } x_1 = y_1,\ x_2 = y_2, \\ 0, & \text{for } x_1 \neq y_1,\ x_2 \neq y_2 \end{cases} \quad \text{on } \Omega_h; \tag{4}$$

$$\frac{\Delta G}{\Delta n} + \overline{a} G = 0$$

on S_h, as a function of (y_1, y_2). Then on Ω_h, $G \geq 0$.

Let $G = W + w$, where W is the h-singular solution (4), studied by S. L. Sobolev (8) and A. Štrom (11), and w is an h-harmonic function with the following boundary conditions on S_h:

$$\frac{\Delta w}{\Delta n} + \overline{a} w = -\left(\frac{\Delta W}{\Delta n} + \overline{a} W\right) = \varphi_1.$$

Then, if r is the distance between (x_1, x_2) and (y_1, y_2), and $(x_1, x_2) \in \Omega_1$, then φ_1 is uniformly bounded as $h \to 0$, since, according to the results of (8), up to a constant,

$$\left| W - \frac{1}{2\pi} \ln r \right| \leq c_1 h r^{-1},$$

and, consequently, w is also uniformly bounded. From the function w on the set Ω_h' we construct an h-conjugate function z:

$$w_{x_1} = z_{x_2}, \qquad w_{x_2} = z_{x_1}; \tag{5}$$

then on S_h' the following equalities hold for it:

$$\frac{\Delta z}{\Delta s'} = -\bar{a}w + \varphi_1.$$

We choose the function z so that

$$\sum_{S'_h} z = 0,$$

then the function z, together with the first difference quotients taken along S'_h, is uniformly bounded on S'_h. We now estimate the other difference quotients of the function z in the boundary layer of width c_2h. To this end we construct a majorant for z, estimating z near the point s_{i_0} lying on the boundary S. Construct a sector D_1, of radius r_0 and with angle $\pi\theta$, with vertex at this point and lying outside Ω; take the pole of the polar coordinate system (r, φ) at the point s_{i_0}. Suppose that the sector D_1 in this coordinate system is given by the equations $0 \leq r \leq r_0$, $-\pi\theta \leq \varphi \leq 0$. Denote the sector $0 \leq r \leq r_0$, $0 \leq \varphi \leq \pi(2-\theta)$ by D_2, and $D_2\Omega$ by D_3, the boundary of the domain D_3 by S_3; it consists of two parts: S_1, a part of the boundary S, and S_2, a part of the circle $r = r_0$. We assume that S_2 consists of one component; this can be achieved by choosing r_0 sufficiently small. Denote D_3S_h by S_{2h}; $D_3S'_h$ by S'_{2h}, and denote by S_{1h} the set of points of Ω_h at distance up to $2h$ from the line $r = r_0$. At the point B construct a barrier. Let $0 \leq \theta < 1$; $M = 2 \max\limits_{S_{1h}+S_{2h}} r^{-1}|z - z(s'_i)|$; note that on S_{1h}

$$|z(s'_i) - z(s_{i_0})| \leq h \sum_{i_0}^{i} \left|\frac{\Delta z}{\Delta s}\right| \leq c' \max_{S_h} (|\bar{a}w| + |\varphi_1|)\,|s_i - s_{i_0}|.$$

Then the subharmonic function Mr will majorize the function $|z - z_{i_0}|$ on the set $S_{1h} + S_{2h}$, and consequently the function q, harmonic in D_2 and coinciding on the boundary of the sector D_2 with the function Mr, will be a majorant for $|z - z_{i_0}|$, but in a neighborhood of $r = 0$

$$|q| \leq c_3 r^d + O(r),$$

where $d = (2-\theta)^{-1}$, i.e., if $(x_1, x_2) \in \Omega_1$, and the points (y'_1, y'_2) and (y''_1, y''_2) are at a distance less than

c_1h from the point s_i, then, using equations (5), we find that

$$|w(x_1, x_2, y'_1, y'_2) - w(x_1, x_2, y''_1, y''_2)| \leq c_4 h^d.$$

It is clear from the proof that the constants M, c_3, c_4 can be chosen independently of h and of the position of the point s_i on the boundary S; then on S_h

$$\left|\frac{\Delta G}{\Delta s}\right| \leq c_5 h^{d-1}.$$

II. We now proceed to estimate the error of the method presented in (4); the error $\eta = U - u$ satisfies the following equations:

$$\Delta_h \eta = h^2 \psi_1 \quad \text{on } \Omega_h;$$

$$\frac{\Delta \eta}{\Delta n} + \bar{a}\eta = \psi_2 \quad \text{on } S_h. \tag{6}$$

The function ψ_2 can be estimated as follows: when replacing the equality

$$V(s_m) - V(s_0) = \int_{s_0}^{s_m} (\varphi - aU)\, dS \tag{7}$$

by the rectangle formula for the approximate boundary values of the function u and of the h-conjugate function v corresponding to it,

$$v(s'_m) - v(s'_0) = \sum_{S_h,\ k=0}^{m} (\varphi - au)\Delta s_k, \tag{8}$$

we make an error of order h. Comparing (6) and (8), we see that, although $\psi_2 = O(1)$,

$$b(s'_m) = h \sum_{S_h,\ k=0}^{m} \psi_2(s'_k) = O(h), \quad m = 0, 1, \dots, N. \tag{9}$$

The quantity ψ_1 can be estimated (^{5},^{6}) as follows:

$$|\psi_1| \leq c_6 h^2 \omega_\alpha(\rho), \tag{10}$$

where $\alpha = p + \gamma - 3$; ρ is the distance to the boundary; $\omega_\alpha(\rho)$ is a function equal to $\min(h^\alpha, \rho^\alpha)$ for $\alpha \leq 0$ and equal to 1 for $\alpha \geq 0$.

Using the Green function constructed above, from the difference analogue of Green' s formula we obtain

$$\eta(x_1, x_2) = -h^2 \left(h^2 \sum_{\Omega_h} G\psi_1 \right) + h \sum_{S_h} G\psi_2.$$

Let $(x_1, x_2) \subset \Omega_1$. To estimate the first term we use the representation of the Green function $G = W + w$ (the nature of the singularity of W was investigated in (8)), as well as inequality (10); then

$$h^2 \left| h^2 \sum_{\Omega_h} G\psi_1 \right| \leq c_7 h^2 \omega_{\alpha+1}(0).$$

We estimate the second term using (9):

$$\left| h \sum_{S_h} G\psi_2 \right| \leq c_8 \left(|Gb|_{k=0} + |Gb|_{k=N} + h \sum_{S_h} \left| \frac{\Delta G}{\Delta n} b \right| \right) \leq c_9 h^d.$$

Thus, if $d_1 = \min((2-\theta)^{-1},\, p+\gamma)$ and $0 \leq \theta < 1$, then in any strictly interior subdomain Ω_1 of the domain Ω

$$|\eta| \leq c_{10} h^{d_1}.$$

III. We note that the difference scheme considered is the only positive scheme known to the author that is suitable for solving both the second and the third boundary-value problems (4); the schemes in the papers (2,9,10) do not possess this property.

Let us present a three-point positive scheme of accuracy h, suitable for solving the third boundary-value problem. Let u_i be the values of U, respectively, at the points A_i, $i = 0, 1, 2$, with coordinates $(0,0)$, $(0, -h_0)$, $(-h_0, 0)$ in the local coordinate system, and let $A_0 \in S_h$. Construct a linear function u coinciding with U at the points A_i. Let $[s^1, s^2]$ be a segment of the boundary S near the point A_0, of length Δs of order $O(h)$; let $\Delta x_1, \Delta x_2$ be its projections onto the axes x_1, x_2, and let (x_1^1, x_2^1), (x_1^2, x_2^2) be the coordinates of its endpoints. Then

$$\int_{s^1}^{s^2} \frac{\partial u}{\partial n}\, dS = \frac{u_0 - u_2}{h_0} \Delta x_2 - \frac{u_0 - u_1}{h_0} \Delta x_1.$$

Applying an approximate formula to the integral and discarding terms of order $O(h^2)$, we obtain

$$\frac{u_0 - u_2}{h_0} \alpha_2 - \frac{u_0 - u_1}{h_0} \alpha_1 + a u_0 = \varphi,$$

where $\alpha_i = \dfrac{\Delta x_i}{\Delta s}$; by a suitable choice of the endpoints of the segment $[s^1, s^2]$ and of the points A_1, A_2 ($h_0 = h,\ h\sqrt{2},\ h\sqrt{5}$, etc.) we ensure that $\alpha_2 \geqslant 0,\ \alpha_1 \leqslant 0$. Analogous constructions can be carried out in the n-dimensional case, and also for nonharmonic functions.

Finally, consider one four-point positive scheme of accuracy h^2. To this end consider one more point $A_3(0, h_0) \in \Omega_h$, and take the points s^1, s^2 on the hyperbola $x_1^2 - x_2^2 = c^2$ and such that $x_1^1, x_1^2, x_2^1 \geqslant 0$, $x_2^2 \leqslant 0$. Construct a harmonic polynomial u of degree 2, coinciding at the points A_i with the values u_i, $i = 0, 1, 2, 3$. Then, repeating the preceding arguments, we obtain

$$\frac{a_1 + a_2}{2} u_0 + \frac{u_0 - u_2}{h_0} \left(\frac{a_1 x_1^1 + a_2 x_1^2}{2} + \alpha_2 \right) + \frac{u_3 + u_1}{2h_0} \left(\frac{a_1 x_2^1 + a_2 x_2^2}{2} - \alpha_1 \right) +$$

$$+ \frac{2u_0 - u_1 - u_3}{h_0} \left(\frac{x_1^1 x_2^1 - x_1^2 x_2^2}{h_0 \Delta s} + \frac{a_1 x_1^1 + a_2 x_1^2}{2} + \frac{1}{2} \alpha_1 \right) = \frac{\varphi_1 + \varphi_2}{2},$$

where $a_j = a(s^j)$, $\varphi_j = \varphi(s^j)$, $j = 1, 2$.

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